

GCSE MATHEMATICS (9-1) — TUTORIALS & WORKED EXAMPLES

Grade 5 Mathematics — Tutorials & Examples

(Foundation & Higher Crossover)

Higher & Foundation Crossover Tutorials, Common Mistake Warnings & Step-by-Step Worked Examples

- Core Crossover Concept Notes & Precision Geometric/Graph Diagrams
- Exam-Focused "Common Mistake" Warnings on every single step
- Model Worked Examples tagged **Calculator** or **Non-Calculator**
- Streamlined Crossover Revision Guide (Tutorials & Worked Examples Only)

Student Name: _____

Class / Teacher: _____

Target Grade: _____



SCAN FOR ONLINE PRACTICE & WORKED SOLUTIONS

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Reverse Percentages

□ CORE CONCEPT & METHOD

BIGGEST EXAM MISTAKE: You cannot just add/subtract the percentage back on!

Case 1: When the Final Amount is given

If a shirt is in a 20% off sale, it is now worth 80% of its normal price (multiplier: 0.80).

If a bill has a 20% tip added, it is now worth 120% of the original (multiplier: 1.20).

Divide the final amount by the decimal multiplier: $\text{Original} = \frac{\text{Final}}{\text{Multiplier}}$.

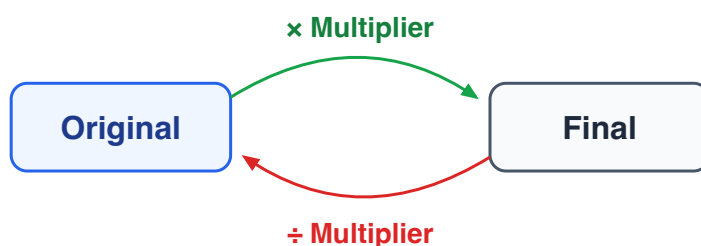
Case 2: When the Change Amount is given

If normal prices are reduced by 25% and the reduction is £12, then 25% is equal to £12.

Divide the change by the percentage decimal:

$\text{Original} = \frac{\text{Change}}{\text{Decimal Percentage}}$.

$$\text{Original} \times \text{Multiplier} = \text{Final}$$



$$\text{Original} = \frac{\text{Final}}{\text{Multiplier}} \quad \text{or} \quad \text{Original} = \frac{\text{Change}}{\text{Percentage}}$$

Common Mistake: Never find the percentage of the sale/final price and add or subtract it — always divide the final amount by the decimal multiplier (e.g. $\div 0.80$ for 20% off, or $\div 1.15$ for a 15% increase).

MODEL WORKED EXAMPLES

EXAMPLE 1

 Calculator

A laptop costs £560 **after** a 20% discount. What was the original price?

STEP-BY-STEP SOLUTION:

1. 20% off means it is 80% of the original (0.80).
2. $\text{Original} \times 0.80 = 560$
3. $\text{Original} = \frac{560}{0.80} = 700$

£700

EXAMPLE 2

 Calculator

A hotel bill is £96 **after** a 20% service charge is added. What was the original bill?

STEP-BY-STEP SOLUTION:

1. 20% added means it is 120% of the original (1.20).
2. $\text{Original} \times 1.20 = 96$
3. $\text{Original} = \frac{96}{1.20} = 80$

£80

EXAMPLE 3

 Calculator

In a sale, normal prices are reduced by 25%. The normal price of a jacket is reduced by £15. Calculate the normal price of the jacket.

STEP-BY-STEP SOLUTION:

1. The reduction of £15 represents 25% of the normal price.
2. $25\% = 0.25$
3. $\text{Normal} = 15 \div 0.25 = 60$

£60

Standard Form (Converting)

□ CORE CONCEPT & METHOD

Standard Form ($A \times 10^n$) is a way of writing very large or very small numbers.

- A MUST be between 1 and 10 (but not including 10).
- n is the number of places the decimal point moves.
- Positive n = large numbers. Negative n = small decimal numbers.

Common Mistake: In $A \times 10^n$, the front number A must satisfy $1 \leq A < 10$ (e.g. 450×10^3 is NOT in standard form — rewrite 450 as 4.5×10^2 to get 4.5×10^5).

MODEL WORKED EXAMPLES

EXAMPLE 1

✎ Non-Calculator

Write 68,000 in standard form.

STEP-BY-STEP SOLUTION:

1. Place a vertical line after the first non-zero digit to make a number between 1 and 10: 6.8
2. Count how many digits are after the line (how many places the decimal moved): 4 places.

$$\begin{array}{c} 6 \mid 8000 \\ 6 \cdot 8 \times 10^4 \end{array}$$

Answer: 6.8×10^4

$$6.8 \times 10^4$$

EXAMPLE 2 **Non-Calculator**

Write 0.00054 in standard form.

STEP-BY-STEP SOLUTION:

1. Place a vertical line after the first non-zero digit: 5.4
2. Count how many places the decimal moved right to reach the line: 4 places.
3. Since it's a small decimal number, the power is negative: -4

$$\begin{array}{c} 0.0005 \mid 4 \\ 5.4 \times 10^{-4} \end{array}$$

Answer: 5.4×10^{-4}

$$5.4 \times 10^{-4}$$

Standard Form (Calculating)

CORE CONCEPT & METHOD

When **Multiplying/Dividing**, handle the normal numbers together, and handle the powers of 10 together using index laws!

Tip: If your final answer has a front number larger than 10 (e.g. 12×10^4), adjust it! Divide the front by 10 and add 1 to the power (1.2×10^5).

Tip: If the front number is less than 1 (e.g. 0.5×10^4), adjust it! Multiply the front by 10 and subtract 1 from the power (5×10^3).

When **Adding/Subtracting**, convert both numbers to the SAME power of 10 first, then add the front numbers.

Common Mistake: Use index laws when multiplying/dividing powers of 10, but when ADDING or SUBTRACTING, convert both numbers to the same power of 10 (or ordinary numbers) first.

MODEL WORKED EXAMPLES

EXAMPLE 1

 **Non-Calculator**

Calculate $(4 \times 10^3) \times (2 \times 10^5)$

STEP-BY-STEP SOLUTION:

1. Multiply the numbers: $4 \times 2 = 8$
2. Add the powers (Index Law): $3 + 5 = 8$

Answer: 8×10^8

8×10^8

EXAMPLE 2 **Non-Calculator**

Calculate $(6 \times 10^4) \times (5 \times 10^3)$

STEP-BY-STEP SOLUTION:

1. Multiply the numbers: $6 \times 5 = 30$

2. Add the powers (Index Law): $4 + 3 = 7$

Initial Answer: 30×10^7

Adjustment needed: 30 is larger than 10.

Divide the front by 10 (making it 3.0) and add 1 to the power.

$$\begin{array}{c} 30 \times 10^7 \\ \downarrow \\ 3 \times 10 \times 10^7 \\ \downarrow \\ 3 \times 10^8 \end{array}$$

Final Answer: 3×10^8

$$3 \times 10^8$$

EXAMPLE 3 **Non-Calculator****Calculate** $(3 \times 10^7) \div (5 \times 10^2)$ **STEP-BY-STEP SOLUTION:**

1. Divide the numbers: $3 \div 5 = 0.6$
2. Subtract the powers (Index Law): $7 - 2 = 5$

Initial Answer: 0.6×10^5 **Adjustment needed:** 0.6 is less than 1.

Multiply the front by 10 (making it 6) and subtract 1 from the power.

$$\begin{array}{c}
 0.6 \times 10^5 \\
 \downarrow \\
 0.6 \times 10 \times 10^4 \\
 \downarrow \\
 6 \times 10^4
 \end{array}$$

Final Answer: 6×10^4

$$6 \times 10^4$$

EXAMPLE 4 **Non-Calculator****Calculate** $(5.1 \times 10^6) + (4 \times 10^5)$ **STEP-BY-STEP SOLUTION:**To add, the powers of 10 **MUST** be the same.

1. Change 4×10^5 to 0.4×10^6
2. Now add the front numbers: $5.1 + 0.4 = 5.5$

Answer: 5.5×10^6

$$5.5 \times 10^6$$

EXAMPLE 5 **Non-Calculator**

Calculate $(7 \times 10^9) - (3 \times 10^8)$

STEP-BY-STEP SOLUTION:

To subtract, the powers of 10 MUST be the same.

1. Change 3×10^8 to 0.3×10^9
2. Now subtract the front numbers: $7 - 0.3 = 6.7$

Answer: 6.7×10^9

$$6.7 \times 10^9$$

Changing the Subject

CORE CONCEPT & METHOD

Changing the subject of a formula is exactly like solving an equation, but instead of getting a number at the end, you get a new formula! Just use the balancing method to get the letter you want on its own.

Common Mistake: Undo operations in reverse BIDMAS order: clear full-width fractions or square roots BEFORE moving terms trapped inside them.

MODEL WORKED EXAMPLES

EXAMPLE 1

 **Non-Calculator**

Make x the subject of the formula: $y = 4x - 5$

STEP-BY-STEP SOLUTION:

1. We want to get x on its own. First, get rid of the -5 by adding 5 to both sides (line up $+5$ under y and under -5):

$$\begin{array}{rcl} y & = & 4x - 5 \\ +5 & & +5 \\ \hline y + 5 & = & 4x \end{array}$$

2. Now, do the opposite of multiplying by 4. Divide both sides by 4:

$$\begin{array}{rcl} y + 5 & = & 4x \\ \div 4 & & \div 4 \\ \hline \frac{y + 5}{4} & = & x \end{array}$$

$$x = \frac{y + 5}{4}$$

EXAMPLE 2 **Non-Calculator**

Make x the subject of the formula: $y = \frac{3x + 5}{2}$

STEP-BY-STEP SOLUTION:

1. When everything is divided by a number, multiply by it first to get rid of the fraction!

$$\begin{array}{rcl} y & = & \frac{3x + 5}{2} \\ \times 2 & & \times 2 \\ \hline 2y & = & 3x + 5 \end{array}$$

2. Subtract 5 from both sides:

$$\begin{array}{rcl} 2y & = & 3x + 5 \\ -5 & & -5 \\ \hline 2y - 5 & = & 3x \end{array}$$

3. Divide both sides by 3:

$$\begin{array}{rcl} 2y - 5 & = & 3x \\ \div 3 & & \div 3 \\ \hline \frac{2y - 5}{3} & = & x \end{array}$$

$$x = \frac{2y - 5}{3}$$

EXAMPLE 3 **Non-Calculator**

Make n the subject of the formula: $m = n^2 + 7$

STEP-BY-STEP SOLUTION:

1. Subtract 7 from both sides to get the n^2 on its own:

$$\begin{array}{rcl} m & = & n^2 + 7 \\ -7 & & -7 \\ \hline m - 7 & = & n^2 \end{array}$$

2. The opposite of squaring is square rooting! Square root both sides:

$$\begin{array}{rcl} m - 7 & = & n^2 \\ \sqrt{} & & \sqrt{} \\ \hline \sqrt{m - 7} & = & n \end{array}$$

$$n = \sqrt{m - 7}$$

EXAMPLE 4 **Non-Calculator**

Make b the subject of the formula: $a = \sqrt{\frac{b+4}{3}}$

STEP-BY-STEP SOLUTION:

1. Square both sides to eliminate the square root:

$$\begin{array}{rcl} a & = & \sqrt{\frac{b+4}{3}} \\ (\cdot)^2 & & (\cdot)^2 \\ \hline a^2 & = & \frac{b+4}{3} \end{array}$$

2. Multiply both sides by 3:

$$\begin{array}{rcl} a^2 & = & \frac{b+4}{3} \\ \times 3 & & \times 3 \\ \hline 3a^2 & = & b+4 \end{array}$$

3. Subtract 4 from both sides:

$$\begin{array}{rcl} 3a^2 & = & b+4 \\ -4 & & -4 \\ \hline 3a^2 - 4 & = & b \end{array}$$

$$b = 3a^2 - 4$$

EXAMPLE 5 **Non-Calculator**

Make b the subject of the formula: $a = \frac{5+c}{b}$

STEP-BY-STEP SOLUTION:

1. Multiply both sides by b to clear the fraction:

$$\begin{array}{rcl} a & = & \frac{5+c}{b} \\ \times b & & \times b \\ \hline ab & = & 5+c \end{array}$$

2. Divide both sides by a :

$$\begin{array}{rcl} ab & = & 5+c \\ \div a & & \div a \\ \hline b & = & \frac{5+c}{a} \end{array}$$

$$b = \frac{5+c}{a}$$

Gradient of a Line

1 Gradient from Equation

□ CORE CONCEPT & METHOD

Equation of a Line

$$y = mx + c$$

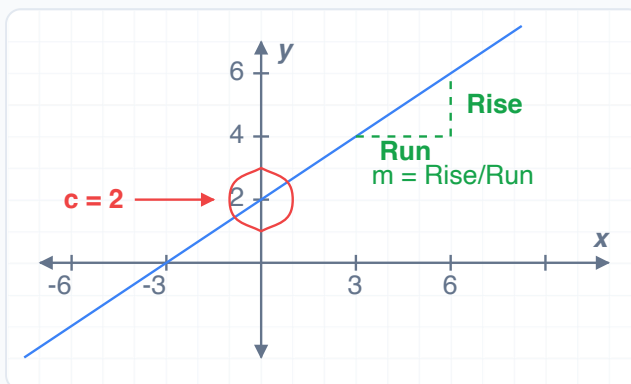
m = Gradient

$$\left(\frac{\text{Rise}}{\text{Run}} \right)$$

c = y-intercept

(where line meets y-axis)

Reminder for Coordinates: Go 'Along the corridor (x), then up the stairs (y)'.



For the graph above:

$$m = \frac{2}{3} \quad c = 2$$

$$y = \frac{2}{3}x + 2$$

Gradient:**Positive (+)****Negative (-)**

Common Mistake: Gradient is Rise $\div$ Run $((y_2 - y_1) \div (x_2 - x_1))$, NEVER change in x divided by change in y — and check that lines sloping downwards left-to-right have a NEGATIVE sign.

MODEL WORKED EXAMPLES**EXAMPLE 1.1** **Non-Calculator**

A line has the equation $y = 6x - 7$.

Write down the gradient of the line and the coordinates of its y -intercept.

STEP-BY-STEP SOLUTION:

Comparing $y = 6x - 7$ to $y = mx + c$:

1. Gradient

The value of m is 6.

2. y -intercept

The value of c is -7 .

This means the line crosses the y -axis at -7 .

So the coordinates are $(0, -7)$.

$$m = 6 \quad y\text{-int} = (0, -7)$$

2 Gradient from Points (Pos) **CORE CONCEPT & METHOD**

To find the gradient between two points (x_1, y_1) and (x_2, y_2) , use the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

(Change in y divided by change in x).

Common Mistake: Always subtract coordinates in the SAME order on top and bottom $(\frac{y_2 - y_1}{x_2 - x_1})$, never mixing $(y_2 - y_1)$ with $(x_1 - x_2)$.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

Non-Calculator

Find the gradient of the line passing through (3, 4) and (6, 19).

STEP-BY-STEP SOLUTION:

$$x_1 = 3, y_1 = 4$$

$$x_2 = 6, y_2 = 19$$

$$m = \frac{19 - 4}{6 - 3} = \frac{15}{3} = 5$$

5

3 Gradient from Points (Neg)

CORE CONCEPT & METHOD

Be very careful with **negative numbers** when subtracting! Use brackets if needed:

$$\frac{y_2 - (y_1)}{x_2 - (x_1)}.$$

Common Mistake: Wrap negative coordinates in brackets when subtracting (e.g. $5 - (-3) = 8$, never 2), and check whether the line slopes uphill (+) or downhill (-).

MODEL WORKED EXAMPLES

EXAMPLE 3.1

Non-Calculator

Find the gradient of the line passing through (-2, -7) and (3, 13).

STEP-BY-STEP SOLUTION:

$$m = \frac{13 - (-7)}{3 - (-2)} = \frac{13 + 7}{3 + 2} = \frac{20}{5} = 4$$

4

EXAMPLE 3.2

Non-Calculator

Find the gradient of the line passing through (-3, 11) and (2, -4).

STEP-BY-STEP SOLUTION:

$$x_1 = -3, y_1 = 11$$

$$x_2 = 2, y_2 = -4$$

$$m = \frac{-4 - 11}{2 - (-3)} = \frac{-15}{2 + 3} = \frac{-15}{5} = -3$$

(Notice that the result is a **negative** gradient, meaning the line slopes downwards).

-3

4 Unknown Coordinate

□ CORE CONCEPT & METHOD

If you are given the gradient but missing a coordinate, set up the gradient formula as an **algebraic equation** and solve it.

Common Mistake: Substitute the known gradient m into $\frac{y_2 - y_1}{x_2 - x_1} = m$ and multiply by $(x_2 - x_1)$ first before solving for k .

MODEL WORKED EXAMPLES

EXAMPLE 4.1

✎ Non-Calculator

A line passes through $(2, 5)$ and $(6, k)$. The gradient is 4. Work out the value of k .

STEP-BY-STEP SOLUTION:

$$\frac{k - 5}{6 - 2} = 4$$

$$\frac{k - 5}{4} = 4$$

$$k - 5 = 16$$

$$k = 21$$

21

5 Equation from a Graph

□ CORE CONCEPT & METHOD

How to Find the Equation of a Line ($y = mx + c$) from a Graph:

1. **Find c (the y -intercept):** Look at where the line crosses the vertical y -axis.

2. **Find m (the gradient):** Pick two clear grid points on the line, draw a right-

angled triangle, and calculate $m = \frac{\text{Rise}}{\text{Run}}$.

(Check the direction: upwards left-to-right is **positive** $+m$; downwards left-to-right is **negative** $-m$).

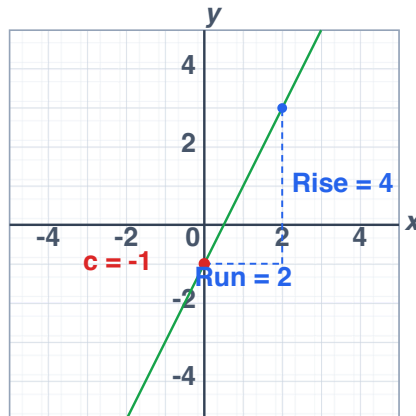
3. **Write the equation:** Substitute your values of m and c into $y = mx + c$.

Common Mistake: Read the x -axis and y -axis scales carefully rather than counting grid squares blindly — one grid square is not always 1 unit.

MODEL WORKED EXAMPLES

EXAMPLE 5.1 **Non-Calculator**

Find the equation of the line shown on the grid.

**STEP-BY-STEP SOLUTION:****1. Find the y -intercept (c):**The line crosses the y -axis at $(0, -1)$, so $c = -1$.**2. Find the gradient (m):**Choosing the points $(0, -1)$ and $(2, 3)$ on the line:

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{4}{2} = 2$$

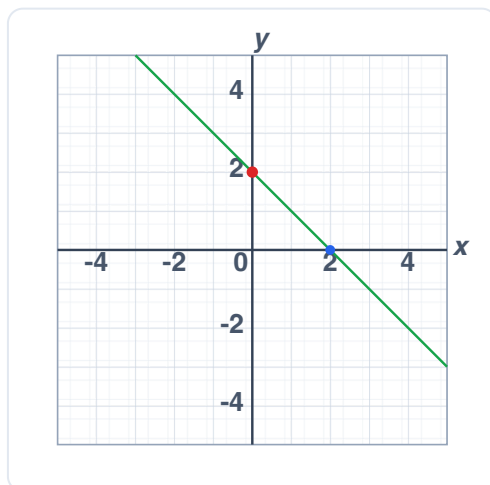
3. Write the equation in the form $y = mx + c$:Substitute $m = 2$ and $c = -1$:

$$y = 2x - 1$$

$$y = 2x - 1$$

EXAMPLE 5.2 **Non-Calculator**

Find the equation of the line shown on the grid.

**STEP-BY-STEP SOLUTION:****1. Find the y -intercept (c):**

The line crosses the y -axis at $(0, 2)$, so $c = 2$.

2. Find the gradient (m):

The line passes through $(0, 2)$ and $(2, 0)$ and slopes downwards:

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{-2}{2} = -1$$

3. Write the equation in the form $y = mx + c$:

Substitute $m = -1$ and $c = 2$:

$$y = -x + 2$$

$$y = -x + 2$$

ALGEBRA — COORDINATE GEOMETRY

Equation of a Line

1 Read m and c

CORE CONCEPT & METHOD

In $y = mx + c$, m is the gradient and c is the y -intercept (where the line crosses the y -axis).

Common Mistake: Rearrange into $y = mx + c$ before reading the gradient m (in $2y - 8x = 4$, divide by 2 to get $y = 4x + 2$, so $m = 4$, not 8 or -8).

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

A straight line has equation $y = 7x - 5$. Write down the y -intercept.

STEP-BY-STEP SOLUTION:

Comparing to $y = mx + c$, the value of c is -5 .

$$-5$$

2 Find Equation from Point

CORE CONCEPT & METHOD

If you know the gradient m and a point (x_1, y_1) , you can find the equation by substituting them into $y - y_1 = m(x - x_1)$ or $y = mx + c$.

Common Mistake: After finding m , substitute BOTH coordinates of a given point (x_1, y_1) into $y = mx + c$ to solve for c — never assume y_1 is the y -intercept unless $x_1 = 0$.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

Non-Calculator

A line has a gradient of 4 and passes through $(3, 19)$. Find the equation of the line.

STEP-BY-STEP SOLUTION:

Using $y = mx + c$:

$$19 = 4(3) + c$$

$$19 = 12 + c$$

$$c = 7$$

Equation is $y = 4x + 7$.

$$y = 4x + 7$$

3 Rearrange Equations

□ CORE CONCEPT & METHOD

If an equation is not in the form $y = mx + c$, use algebra to rearrange it so that y is isolated on its own.

Common Mistake: When dividing an equation like $3y = 15x + 12$ by 3 to isolate y , divide EVERY term on the right-hand side by 3, not just the x -term.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

✎ Non-Calculator

Find the gradient of the line $3y - 15x = 12$.

STEP-BY-STEP SOLUTION:

To find the gradient, we must first rearrange the equation into the form:

$$y = mx + c$$

$$3y - 15x = 12$$

Add $15x$ to both sides:

$$3y = 15x + 12$$

Divide everything by 3:

$$y = 5x + 4$$

Now the equation is in the correct form, we can see the gradient (m) is 5.

$$\text{Gradient} = 5$$

4 Parallel Equations

□ CORE CONCEPT & METHOD

Parallel lines have the exact same gradient! If a line is parallel to $y = 5x - 2$, its gradient is also 5.

Common Mistake: Parallel lines share the EXACT same gradient m , NOT the same y -intercept c — substitute the new point (x, y) into $y = mx + c$ to find the new c .

MODEL WORKED EXAMPLES

EXAMPLE 4.1

✎ Non-Calculator

Find the equation of a line parallel to $y = 6x - 1$ passing through $(0, 9)$.

STEP-BY-STEP SOLUTION:

Gradient $m = 6$.

Passes through $(0, 9)$, so y -intercept $c = 9$.

Equation: $y = 6x + 9$.

$$y = 6x + 9$$

EXAMPLE 4.2 **Non-Calculator**

Find the equation of a line parallel to $y = 4x - 7$ passing through $(3, 17)$.

STEP-BY-STEP SOLUTION:

Parallel lines have the same gradient, so $m = 4$.

This gives us part of the equation: $y = 4x + c$.

Substitute the x and y values from the point $(3, 17)$ to find c :

$$17 = 4(3) + c$$

$$17 = 12 + c$$

$$c = 5$$

Equation: $y = 4x + 5$.

$$y = 4x + 5$$

5 Equation from a Graph **CORE CONCEPT & METHOD****Finding the Equation of a Line ($y = mx + c$) from a Graph:**

- Find c (the y -intercept):** Read the value where the line crosses the vertical y -axis.
- Find m (the gradient):** Pick two clear grid points on the line, draw a right-angled triangle, and calculate $m = \frac{\text{Rise}}{\text{Run}}$.
- Substitute m and c :** Write the final equation as $y = mx + c$.

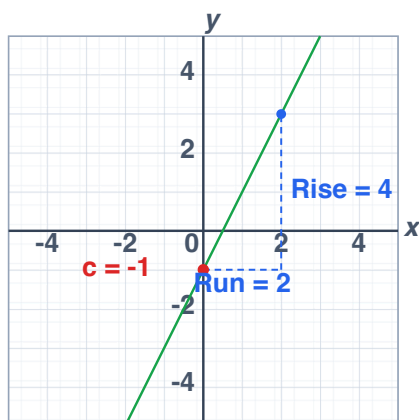
Common Mistake: Pick two points where the line crosses exact grid intersections to calculate

$m = \frac{\text{Rise}}{\text{Run}}$, and attach a minus sign if the line falls from left to right.

MODEL WORKED EXAMPLES

EXAMPLE 5.1 **Non-Calculator**

Find the equation of the line shown on the grid.

**STEP-BY-STEP SOLUTION:****1. Find the y -intercept (c):**The line crosses the y -axis at $(0, -1)$, so $c = -1$.**2. Find the gradient (m):**Choosing the points $(0, -1)$ and $(2, 3)$ on the line:

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{4}{2} = 2$$

3. Write the equation in the form $y = mx + c$:Substitute $m = 2$ and $c = -1$:

$$y = 2x - 1$$

$$y = 2x - 1$$

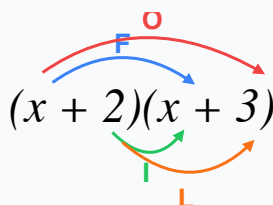
Expanding & Factorising Quadratics

1 Expanding Double Brackets

📖 CORE CONCEPT & METHOD

To expand double brackets, everything in the first bracket multiplies everything in the second!

Method 1: FOIL (Firsts, Outsides, Insides, Lasts)



Firsts: $x \times x = x^2$

Outsides: $x \times +3 = +3x$

Insides: $+2 \times x = +2x$

Lasts: $+2 \times +3 = +6$

$$= x^2 + 3x + 2x + 6$$

$$= x^2 + 5x + 6$$

Method 2: The Grid Method

×	x	$+3$
x	x^2	$+3x$
$+2$	$+2x$	$+6$

Finally, collect the x terms diagonally in the middle:

$$+3x + 2x = +5x$$

So the final expanded expression is:

$$x^2 + 5x + 6$$

Common Mistake: When squaring a single bracket like $(a - 6)^2$, write it out twice as $(a - 6)(a - 6) = a^2 - 12a + 36$ — never just square the two terms to get $a^2 + 36$.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

 Non-CalculatorExpand and simplify $(x + 4)(x + 5)$

STEP-BY-STEP SOLUTION:

Firsts: $x \times x = x^2$ Outsides: $x \times 5 = 5x$ Insides: $4 \times x = 4x$ Lasts: $4 \times 5 = 20$ Put it together: $x^2 + 5x + 4x + 20$ Collect the middle terms ($5x + 4x$):

$$x^2 + 9x + 20$$

EXAMPLE 1.2

 Non-CalculatorExpand and simplify $(x - 6)(x + 4)$

STEP-BY-STEP SOLUTION:

Firsts: $x \times x = x^2$ Outsides: $x \times +4 = +4x$ Insides: $-6 \times x = -6x$ Lasts: $-6 \times +4 = -24$ Put it together: $x^2 + 4x - 6x - 24$ Collect the middle terms ($4x - 6x$):

$$x^2 - 2x - 24$$

EXAMPLE 1.3 **Non-Calculator****Expand and simplify** $(3x - 2)(2x + 5)$ **STEP-BY-STEP SOLUTION:****Method 1: FOIL****Firsts:** $3x \times 2x = 6x^2$ **Outsides:** $3x \times +5 = +15x$ **Insides:** $-2 \times 2x = -4x$ **Lasts:** $-2 \times +5 = -10$ **Method 2: The Grid Method**

×	$2x$	$+5$
$3x$	$6x^2$	$+15x$
-2	$-4x$	-10

Both methods give us: $6x^2 + 15x - 4x - 10$.Collect the middle terms ($+15x - 4x = +11x$):

$$6x^2 + 11x - 10$$

EXAMPLE 1.4 **Non-Calculator****Expand and simplify** $(a - 6)^2$ **STEP-BY-STEP SOLUTION:**Remember that $(a - 6)^2$ means $(a - 6)(a - 6)$!**Firsts:** $a \times a = a^2$ **Outsides:** $a \times -6 = -6a$ **Insides:** $-6 \times a = -6a$ **Lasts:** $-6 \times -6 = +36$ Put it together: $a^2 - 6a - 6a + 36$ Collect the middle terms ($-6a - 6a = -12a$):

$$a^2 - 12a + 36$$

EXAMPLE 1.5 **Non-Calculator****Expand and simplify** $(x + 5)(x - 5)$ **STEP-BY-STEP SOLUTION:**This is the **Difference of Two Squares!****Firsts:** $x \times x = x^2$ **Outsides:** $x \times -5 = -5x$ **Insides:** $5 \times x = +5x$ **Lasts:** $5 \times -5 = -25$ Notice the middle terms cancel out: $+5x - 5x = 0!$

Answer:

$$x^2 - 25$$

2 Factorising Quadratics **CORE CONCEPT & METHOD**To put a quadratic like $x^2 + bx + c$ back into double brackets, you need to play a number puzzle.Find two numbers that **multiply** to make the end number (c), and **add** to make the middle number (b)!**Common Mistake:** Check the signs of your factor pair by multiplying AND adding: for $x^2 - 4x - 21$, the numbers must multiply to -21 and add to -4 (-7 and $+3$, never $+7$ and -3).**MODEL WORKED EXAMPLES****EXAMPLE 2.1** **Non-Calculator****Factorise** $x^2 + 7x + 10$ **STEP-BY-STEP SOLUTION:**We need two numbers that **Multiply** to $+10$ and **Add** to $+7$. Let's list the factors of 10!

Multiply = $+10$	Add = $+7$
1×10	$1 + 10 = 11 \times$
2×5	$2 + 5 = +7 \checkmark$

The winning numbers are $+2$ and $+5$. Put them into brackets!

$$(x + 2)(x + 5)$$

EXAMPLE 2.2 **Non-Calculator****Factorise** $x^2 - 3x - 18$ **STEP-BY-STEP SOLUTION:**We need two numbers that **Multiply** to -18 and **Add** to -3 .

(Hint: To multiply to a negative, one number must be positive and one negative!)

Multiply = -18	Add = -3
-3×6	$-3 + 6 = 3 \times$
3×-6	$3 + -6 = -3 \checkmark$

The winning numbers are $+3$ and -6 . Put them into brackets!

$$(x + 3)(x - 6)$$

EXAMPLE 2.3 **Non-Calculator****Factorise** $x^2 - 7x + 10$ **STEP-BY-STEP SOLUTION:**We need two numbers that **Multiply** to $+10$ and **Add** to -7 .(Hint: To multiply to a positive but add to a negative, **both** numbers must be negative!)

Multiply = $+10$	Add = -7
-1×-10	$-1 + -10 = -11 \times$
-2×-5	$-2 + -5 = -7 \checkmark$

The winning numbers are -2 and -5 . Put them into brackets!

$$(x - 2)(x - 5)$$

EXAMPLE 2.4 **Non-Calculator****Factorise** $x^2 - 64$ **STEP-BY-STEP SOLUTION:**This is the **Difference of Two Squares**! Notice there is no middle x term (it has a value of 0).We need two numbers that multiply to -64 and add to 0.The square root of 64 is 8, so the numbers are $+8$ and -8 .

$$(x - 8)(x + 8)$$

EXAMPLE 2.5 **Non-Calculator****Factorise** $x^2 + 12x + 36$ **STEP-BY-STEP SOLUTION:**

We need two numbers that multiply to $+36$ and add to $+12$.

$6 \times 6 = 36$ and $6 + 6 = 12$.

Since both brackets are identical, $(x + 6)(x + 6)$, this is a **perfect square**!

$$(x + 6)(x + 6)$$

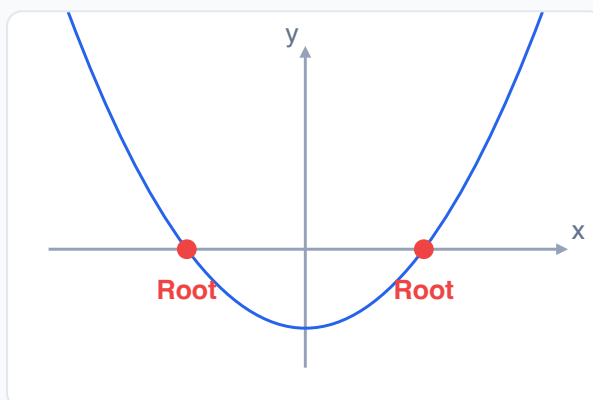
Solving Quadratics by Factorising

1 Introduction to Quadratics

□ CORE CONCEPT & METHOD

Before we learn *how* to solve quadratics, we need to understand *what* we are actually doing!

Solving a quadratic equation like $x^2 + 2x - 8 = 0$ means finding the exact values of x where the quadratic curve crosses the x -axis. These crossing points are called the **roots** or **solutions**.



There are 3 main ways to find these roots:

- **Factorisation:** The quickest method, but it doesn't work for every equation.
- **Completing the Square (Higher):** Always works, and also reveals the turning point!
- **The Quadratic Formula (Higher):** Always works, perfect for nasty decimal answers.

Common Mistake: Make sure the quadratic equals ZERO ($ax^2 + bx + c = 0$) before factorising, and flip the signs inside the brackets when writing down the roots.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✎ Non-Calculator

A quadratic curve $y = x^2 - 3x - 10$ crosses the x -axis at $(-2, 0)$ and $(5, 0)$. Write down the two roots of the equation $x^2 - 3x - 10 = 0$.

STEP-BY-STEP SOLUTION:

1. The roots of $ax^2 + bx + c = 0$ are the x -coordinates where the curve crosses the x -axis ($y = 0$).
2. Reading from $(-2, 0)$ and $(5, 0)$, the roots are $x = -2$ and $x = 5$.

2 Solving by Factorisation

□ CORE CONCEPT & METHOD

To solve a quadratic equation like $x^2 + 5x + 6 = 0$, you should always try to **factorise** it first!

1. Factorise it into double brackets: $(x + 2)(x + 3) = 0$

2. For two things to multiply and make 0, one of them **must be 0!**

So, either $x + 2 = 0$ (meaning $x = -2$) or $x + 3 = 0$ (meaning $x = -3$).

Shortcut: The solutions are just the opposite signs of the numbers in the brackets!

Common Mistake: Do not stop at $(x + p)(x + q) = 0$ — set each bracket equal to zero separately so $(x - 8)(x + 7) = 0$ gives $x = 8$ and $x = -7$ (opposite signs!).

MODEL WORKED EXAMPLES

EXAMPLE 2.1

Solve $x^2 + 8x + 15 = 0$

☑ Non-Calculator

STEP-BY-STEP SOLUTION:

- Factorise: $(x + 3)(x + 5) = 0$
- Opposite signs: $x = -3$ or $x = -5$

$$x = -3, x = -5$$

EXAMPLE 2.2

Solve $x^2 - 3x - 18 = 0$

☑ Non-Calculator

STEP-BY-STEP SOLUTION:

- Factorise: $(x - 6)(x + 3) = 0$
- Opposite signs: $x = 6$ or $x = -3$

$$x = 6, x = -3$$

EXAMPLE 2.3**Non-Calculator****(a) Factorise** $a^2 + 4a - 32$ **(b) Hence, solve** $a^2 + 4a - 32 = 0$ **STEP-BY-STEP SOLUTION:**1. Factorise: Find two numbers that multiply to -32 and add to 4: 8 and -4 .Brackets: $(a + 8)(a - 4)$ 2. Solve $(a + 8)(a - 4) = 0$:

$$a + 8 = 0 \implies a = -8$$

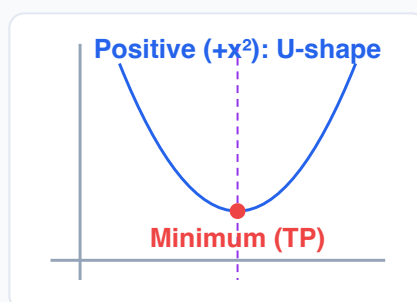
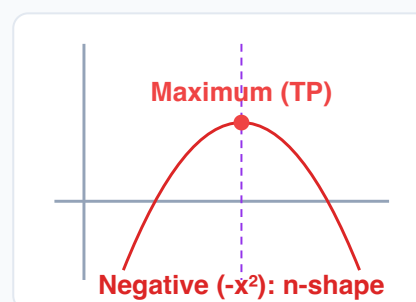
$$a - 4 = 0 \implies a = 4$$

$$a = -8, a = 4$$

3 Quadratic Graphs & Turning Points**CORE CONCEPT & METHOD****1. Recognizing the Shape (Parabola)**

Every quadratic graph $y = ax^2 + bx + c$ forms a smooth symmetrical curve called a **parabola**:

- **Positive x^2 ($a > 0$):** Forms a **U-shape** (a "smiley face"), where the turning point is at the very bottom (**minimum point**).
- **Negative x^2 ($a < 0$, e.g. $y = -x^2 + \dots$):** Forms an **n-shape** (a "sad face"), where the turning point is at the very top (**maximum point**).

**2. Finding the Turning Point from a Graph**

Read the coordinates (x, y) of the lowest point (minimum) or highest point (maximum) directly off the plotted quadratic graph.

Symmetry Tip: Quadratic graphs are symmetrical! The turning point is always **halfway between the two roots** (where the curve crosses the x -axis).

3. Line of Symmetry

The **line of symmetry** is a vertical mirror line that passes straight through the turning point. Its equation is always:

$$x = [\text{the } x\text{-coordinate of the turning point}]$$

(For example, if the turning point is at $(2, -4)$, the line of symmetry is $x = 2$).

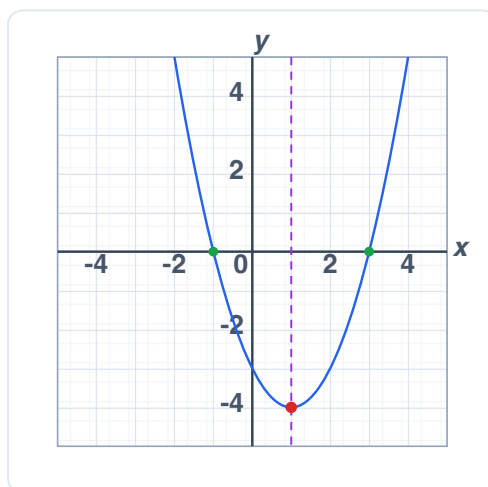
Common Mistake: Write turning points as coordinates (x, y) , roots as x -values where $y = 0$, and the line of symmetry as a full vertical line equation $x = h$ (never just a number).

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 **Non-Calculator**

The diagram shows the graph of $y = x^2 - 2x - 3$.



- Write down the coordinates of the turning point.
- Write down the equation of the line of symmetry.
- Write down the roots of $x^2 - 2x - 3 = 0$.

STEP-BY-STEP SOLUTION:

(a) Turning point:

Looking at the bottom of the U-shape (minimum point), the coordinates are $(1, -4)$.

(b) Line of symmetry:

The vertical dashed mirror line passes through the turning point at $x = 1$, so the equation is $x = 1$.

(c) Roots:

The curve crosses the x -axis at $x = -1$ and $x = 3$.

$$(a) (1, -4) \quad (b) x = 1 \quad (c) x = -1, 3$$

Linear Simultaneous Equations

1 Linear Simultaneous

▣ CORE CONCEPT & METHOD

Simultaneous equations mean we have two equations and two missing letters (x and y). We need to find the numbers that work for **both** equations at the same time!

The Elimination Method: Make the number of x 's or y 's the same in both equations. If the signs are the SAME, SUBTRACT the equations. If the signs are DIFFERENT, ADD the equations.

- **Same Sign Subtract (SSS)**
- **Different Sign Add (DSA)**

Common Mistake: Remember SSS / DSA for the matching variable: if the signs are the SAME, SUBTRACT the equations; if the signs are DIFFERENT (+ and -), ADD the equations.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✎ Non-Calculator

Solve:

$$2x + y = 11$$

$$x - y = 4$$

STEP-BY-STEP SOLUTION:

1. The y 's match! One is $+y$ and one is $-y$. The signs are DIFFERENT, so we ADD the equations together:

$$2x + y = 11$$

$$+ \quad + \quad +$$

$$x - y = 4$$

$$\hline 3x \quad \quad = 15$$

$$x = 5$$

Reminder:

$$+y + -y = y - y = 0$$

2. Substitute $x = 5$ back into the first equation:

$$2(5) + y = 11$$

$$10 + y = 11$$

$$y = 1$$

$$x = 5, y = 1$$

EXAMPLE 1.2 **Non-Calculator****Solve:**

$$4x - 2y = 14$$

$$x - 2y = 2$$

STEP-BY-STEP SOLUTION:

1. The $2y$'s match! Both are $-2y$. Because the signs are the SAME, we SUBTRACT the bottom equation from the top one:

$$4x - 2y = 14$$

$$\begin{array}{r}
 x - 2y = 14 \\
 \underline{x - 2y = 2} \\
 3x = 12
 \end{array}$$

$$x = 4$$

Reminder:

$$-2y - -2y = -2y + 2y = 0$$

2. Substitute $x = 4$ back into the second equation:

$$(4) - 2y = 2$$

$$-2y = -2$$

$$y = 1$$

$$x = 4, y = 1$$

EXAMPLE 1.3 **Non-Calculator****Solve:**

$$2x + 3y = 19$$

$$3x - 2y = 9$$

STEP-BY-STEP SOLUTION:

1. Neither the x nor the y coefficients match. Let's make the y coefficients match. The lowest common multiple of 3 and 2 is 6.

Multiply the first equation by 2:

$$4x + 6y = 38$$

Multiply the second equation by 3:

$$9x - 6y = 27$$

2. Now the y 's match ($+6y$ and $-6y$)! The signs are DIFFERENT, so we ADD:

$$\begin{array}{r}
 4x + 6y = 38 \\
 + \quad + \quad + \\
 9x - 6y = 27 \\
 \hline
 13x \qquad = 65 \\
 \\
 x = 5
 \end{array}$$

Reminder:

$$+6y + -6y = 6y - 6y = 0$$

3. Substitute $x = 5$ back into the first equation:

$$2(5) + 3y = 19$$

$$10 + 3y = 19$$

$$3y = 9$$

$$y = 3$$

$$x = 5, y = 3$$

EXAMPLE 1.4 **Non-Calculator****Solve:**

$$2x = 8 + y$$

$$x + 3y = 11$$

STEP-BY-STEP SOLUTION:

1. The first equation is not in standard form ($ax + by = c$). Rearrange it by subtracting y from both sides:

$$2x - y = 8$$

$$x + 3y = 11$$

2. Multiply the second equation by 2 to match the $2x$ terms:

$$2x + 6y = 22$$

$$2x - y = 8$$

3. Both $2x$ terms are positive. Subtract the bottom equation from the top:

$$6y - (-y) = 22 - 8 \implies 7y = 14 \implies y = 2$$

4. Substitute $y = 2$ into $x + 3y = 11$:

$$x + 3(2) = 11 \implies x + 6 = 11 \implies x = 5$$

$$x = 5, y = 2$$

2 Worded Simultaneous **CORE CONCEPT & METHOD**

Sometimes simultaneous equations are hidden inside word problems! You need to translate the English sentences into algebra first.

- **Step 1:** Choose two letters for the unknown items (e.g. x and y).
- **Step 2:** Write an equation for each sentence.
- **Step 3:** Solve them using the elimination method!

Common Mistake: When multiplying an equation to match coefficients, multiply EVERY term including the right-hand constant, and check your final pair (x, y) in BOTH original contexts.

MODEL WORKED EXAMPLES

 **Calculator**
EXAMPLE 2.1

4 croissants and 3 muffins cost £5.30.

2 croissants and 3 muffins cost £3.70.

Find the cost of a croissant and a muffin.

STEP-BY-STEP SOLUTION:

Let x be the cost of a croissant, and y be the cost of a muffin.

1. We form two equations:

$$4x + 3y = 5.30$$

$$2x + 3y = 3.70$$

2. The $3y$'s match and have the SAME signs, so we SUBTRACT the bottom equation from the top one:

$$4x + 3y = 5.30$$

$$\begin{array}{r} + = 5.30 \\ - + = 3.70 \\ \hline 2x = 1.60 \end{array}$$

Reminder:

$$3y - 3y = 0$$

$$x = 0.80$$

3. Substitute $x = 0.80$ back into Equation 2:

$$2(0.80) + 3y = 3.70$$

$$1.60 + 3y = 3.70$$

$$3y = 2.10$$

$$y = 0.70$$

$$\text{Croissant} = \text{£}0.80$$

$$\text{Muffin} = \text{£}0.70$$

Solving Equations Graphically

1 The Intersection Concept

□ CORE CONCEPT & METHOD

If you draw two straight line equations on the same graph, they will cross each other at exactly one point.

The (x, y) coordinate of that intersection is the **solution** to the simultaneous equations!

Common Mistake: Give both the x and y values of the intersection point — and when solving algebraically, set the two y -expressions equal to each other ($m_1x + c_1 = m_2x + c_2$).

MODEL WORKED EXAMPLES

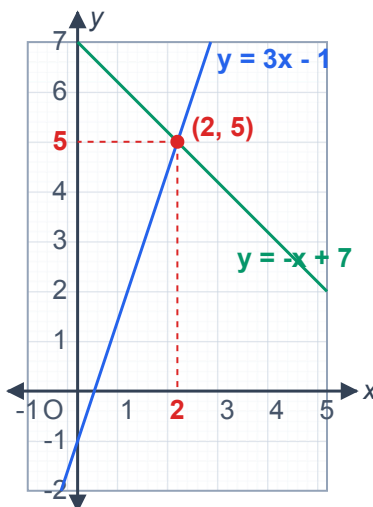
EXAMPLE 1.1

✎ Non-Calculator

By drawing the graphs of $y = 3x - 1$ and $y = -x + 7$, find the solution to the simultaneous equations.

STEP-BY-STEP SOLUTION:

1. Plot both straight lines on the coordinate grid:



2. Find where the two lines cross: they intersect at $(2, 5)$.

3. Therefore, the solution is $x = 2, y = 5$.

2 Finding Intersections Algebraically

CORE CONCEPT & METHOD

If you have two lines like $y = 3x + 1$ and $y = 2x + 4$, you can find where they intersect without drawing them!

Since both equal y , set them equal to each other:

$$3x + 1 = 2x + 4$$

Then solve for x , and substitute back to find y .

Common Mistake: After solving $m_1x + c_1 = m_2x + c_2$ for x , always substitute x back into one of the line equations to find y and write the intersection as coordinates (x, y) .

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 Non-Calculator

The equations of two straight lines are $y = 5x - 7$ and $y = 2x + 5$.
Work out the coordinates of the point where the lines intersect.

STEP-BY-STEP SOLUTION:

1. Set the expressions for y equal to each other:

$$5x - 7 = 2x + 5$$

2. Subtract $2x$ from both sides:

$$3x - 7 = 5$$

3. Add 7 to both sides:

$$3x = 12 \implies x = 4$$

4. Substitute $x = 4$ into either equation:

$$y = 2(4) + 5 = 13$$

5. Coordinates of intersection: $(4, 13)$.

Quadratic, Cubic & Reciprocal Graphs

1 Quadratic Graphs: Tables, Turning Points & Roots

□ CORE CONCEPT & METHOD

Quadratic Graphs: Tables, Turning Points & Roots

Every quadratic function $y = ax^2 + bx + c$ graphs as a smooth, symmetrical curve called a **parabola** (U-shaped if $a > 0$, or $\cap$ -shaped if $a < 0$).

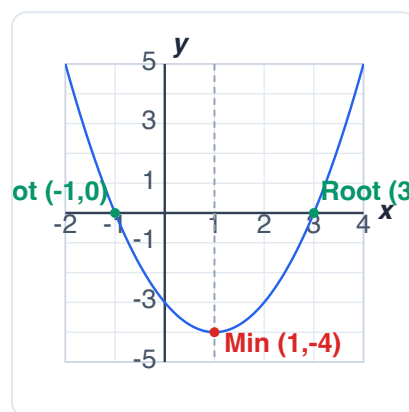
1. Squaring Negatives & Table Symmetry

For $y = x^2 - 2x - 3$ at $x = -2$:

$$y = (-2)^2 - 2(-2) - 3 = +4 + 4 - 3 = 5$$

- **Always put brackets around negative x -values:** $(-3)^2 = +9$, never -9 !
- The y -values repeat symmetrically on either side of the turning point.

2. Turning Points, Roots & $y = k$



- **Turning Point (Vertex):** lowest point $(1, -4)$ on a U-parabola (or highest point on a $\cap$ -parabola).
- **Roots of $ax^2 + bx + c = 0$:** the x -values where the curve crosses the x -axis ($y = 0$).
- **Solving $ax^2 + bx + c = k$:** read the x -coordinates where the curve crosses the horizontal line $y = k$.

Common Mistake: Always wrap negative x -values in brackets when squaring $((-2)^2 = +4$, never -4), and join plotted points with a smooth curve rather than straight ruler segments.

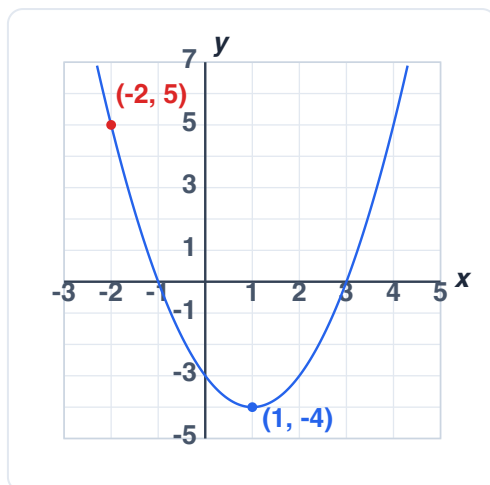
MODEL WORKED EXAMPLES

EXAMPLE 1.1

 Non-Calculator

Complete the table of values for $y = x^2 - 2x - 3$, and find the missing value k when $x = -2$.

x	-2	-1	0	1	2	3	4
y	k	0	-3	-4	-3	0	5



STEP-BY-STEP SOLUTION:

1. Substitute $x = -2$ carefully into $y = x^2 - 2x - 3$ using brackets:

$$k = (-2)^2 - 2(-2) - 3$$

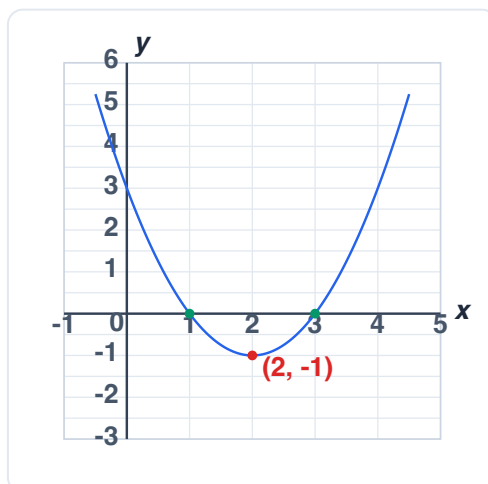
$$k = 4 + 4 - 3 = 5$$

2. Notice how $k = 5$ matches the symmetrical value $y = 5$ at $x = 4$.

5

EXAMPLE 1.2 **Non-Calculator**

The graph of $y = x^2 - 4x + 3$ is drawn on the coordinate grid below.



Write down the coordinates of the minimum turning point of the graph.

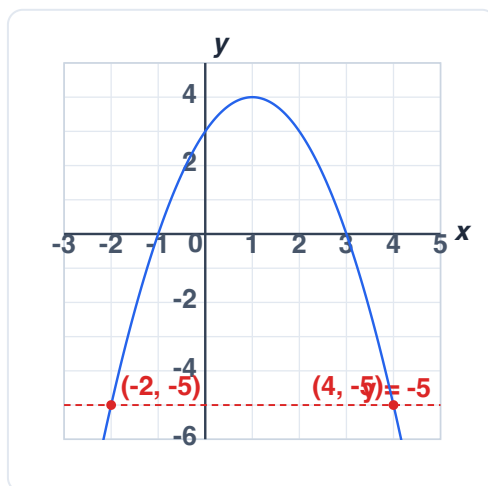
STEP-BY-STEP SOLUTION:

1. Locate the lowest point (vertex) at the bottom of the U-shaped parabola.
2. Reading from the axes, this point lies at $x = 2$ and $y = -1$.
3. Therefore, the minimum turning point is $(2, -1)$. (The roots where $y = 0$ are $x = 1$ and $x = 3$.)

$(2, -1)$

EXAMPLE 1.3 **Non-Calculator**

The graph of $y = 3 + 2x - x^2$ is plotted on the grid below.



Use the graph to find the positive solution to the equation $3 + 2x - x^2 = -5$.

STEP-BY-STEP SOLUTION:

1. To solve $3 + 2x - x^2 = -5$ using the graph of $y = 3 + 2x - x^2$, draw the horizontal line $y = -5$.
2. The line $y = -5$ intersects the parabola at $(-2, -5)$ and $(4, -5)$.
3. The positive solution is $x = 4$.

4

2 Cubic & Reciprocal Graphs

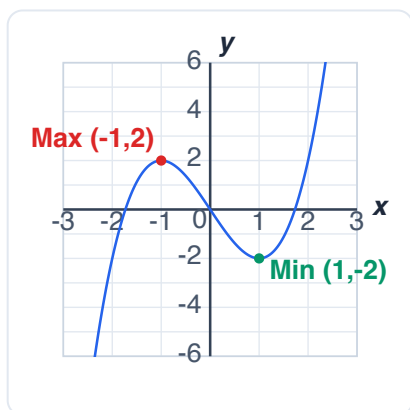
CORE CONCEPT & METHOD

Cubic & Reciprocal Graphs

Understand how cubing negative numbers shapes S-curves and why dividing by x creates two hyperbola branches separated by asymptotes.

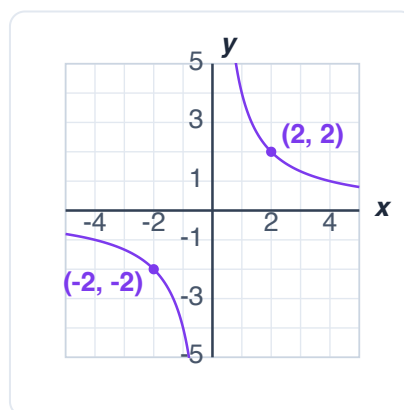
1. Cubic Graphs (

$$y = ax^3 + bx^2 + cx + d)$$



- **Cubing a negative stays negative:**
 $(-2)^3 = -8$.
- A $+x^3$ curve starts in the bottom-left and rises to the top-right, often with a local maximum and local minimum turning point.

2. Reciprocal Graphs ($y = \frac{k}{x}$)



- Because division by 0 is undefined, $y = \frac{k}{x}$ never touches the axes: $x = 0$ and $y = 0$ are **asymptotes**.
- Every point (x, y) on $y = \frac{k}{x}$ satisfies $x \times y = k$.

Common Mistake: Cubing a negative stays negative ($(-2)^3 = -8$), and for reciprocal graphs $y = \frac{k}{x}$, NEVER connect the two separate branches across the vertical asymptote $x = 0$.

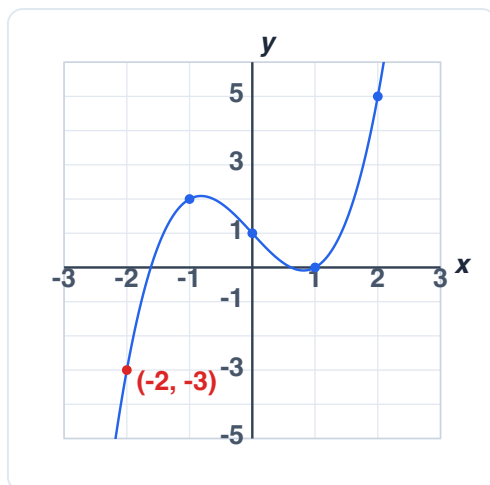
MODEL WORKED EXAMPLES

EXAMPLE 2.1

☒ Non-Calculator

Complete the table of values for $y = x^3 - 2x + 1$, finding the value of k when $x = -2$.

x	-2	-1	0	1	2
y	k	2	1	0	5



STEP-BY-STEP SOLUTION:

1. Substitute $x = -2$ into $y = x^3 - 2x + 1$:

$$k = (-2)^3 - 2(-2) + 1$$

2. Since $(-2)^3 = -8$ and $-2(-2) = +4$:

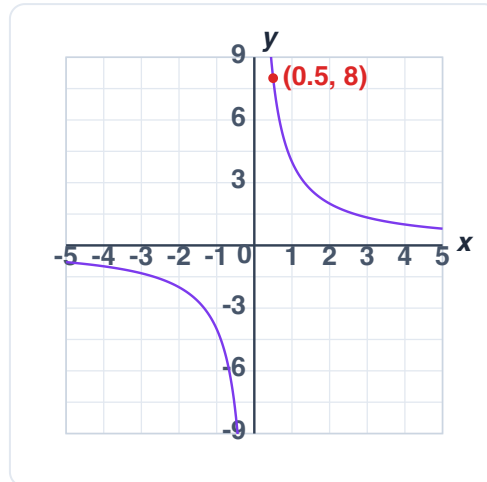
$$k = -8 + 4 + 1 = -3$$

−3

EXAMPLE 2.2**Non-Calculator**

The table and graph below show $y = \frac{4}{x}$ for $-4 \leq x \leq 4$ ($x \neq 0$).

x	-4	-2	-1	-0.5	0.5	1	2	4
y	-1	-2	-4	-8	k	4	2	1



Work out the value of k when $x = 0.5$.

STEP-BY-STEP SOLUTION:

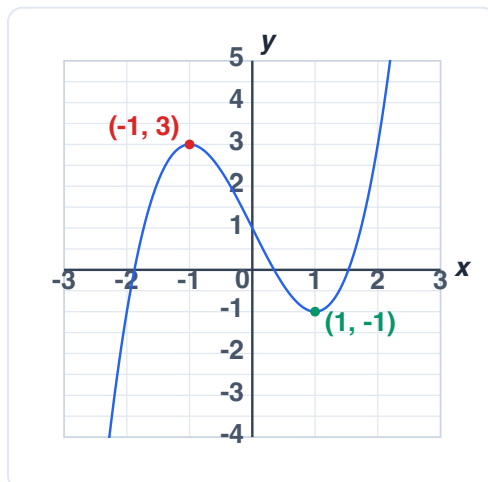
1. Substitute $x = 0.5$ into $y = \frac{4}{x}$:

$$k = \frac{4}{0.5} = 8$$

8

EXAMPLE 2.3 **Non-Calculator**

The graph of $y = x^3 - 3x + 1$ is drawn on the grid below.



Write down the coordinates of the local maximum turning point of the curve.

STEP-BY-STEP SOLUTION:

1. The local maximum turning point is the peak on the left side of the S-curve.
2. Reading from the grid, this peak is at $(-1, 3)$.

$(-1, 3)$

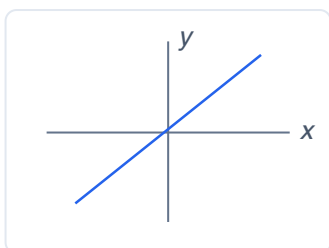
3 Recognising & Matching Graph Shapes

CORE CONCEPT & METHOD

Visual Gallery: The 8 Essential GCSE Graph Shapes

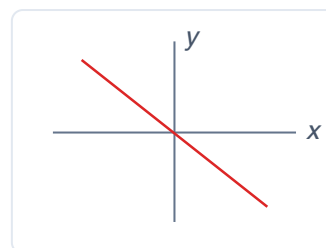
Recognise the shape of any graph immediately from the highest power of x and the sign in front of it.

1. Linear (+x): $y = mx + c$ ($m > 0$)

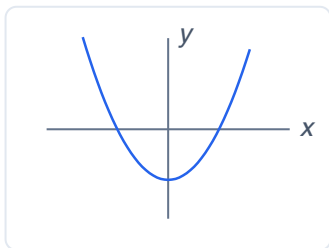


Straight line sloping upwards left-to-right.

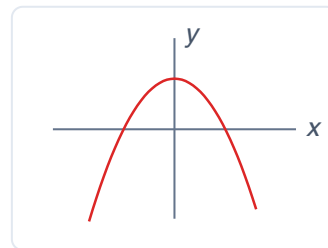
2. Linear (-x): $y = mx + c$ ($m < 0$)



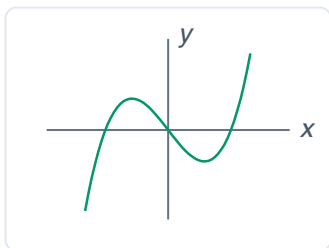
Straight line sloping downwards left-to-right.

3. Quadratic ($+x^2$): $y = ax^2 + \dots$ ($a > 0$)

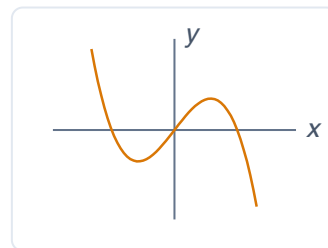
U-shaped parabola with a minimum vertex.

4. Quadratic ($-x^2$): $y = -ax^2 + \dots$ 

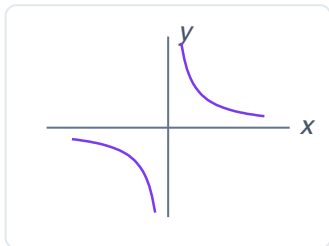
∩-shaped parabola with a maximum vertex.

5. Cubic ($+x^3$): $y = x^3 + \dots$ 

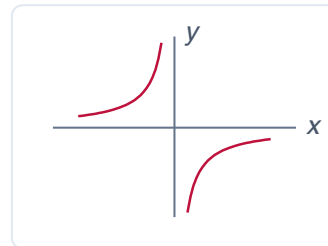
S-curve rising from bottom-left to top-right.

6. Cubic ($-x^3$): $y = -x^3 + \dots$ 

S-curve falling from top-left to bottom-right.

7. Reciprocal ($+\frac{k}{x}$): $k > 0$ 

Two hyperbola branches in Quadrants 1 & 3.

8. Reciprocal ($-\frac{k}{x}$): $k < 0$ 

Two hyperbola branches in Quadrants 2 & 4.

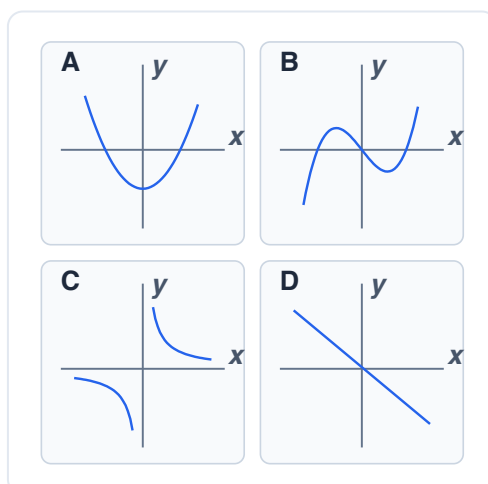
Common Mistake: Check the sign of the highest power first: $+x^2$ is U-shaped while $-x^2$ is ∩-shaped; $+x^3$ rises overall left-to-right while $-x^3$ falls left-to-right.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 Non-Calculator

Four sketched graphs A , B , C , and D are shown below.



Write down the letter of the graph that represents $y = x^3 - 3x$.

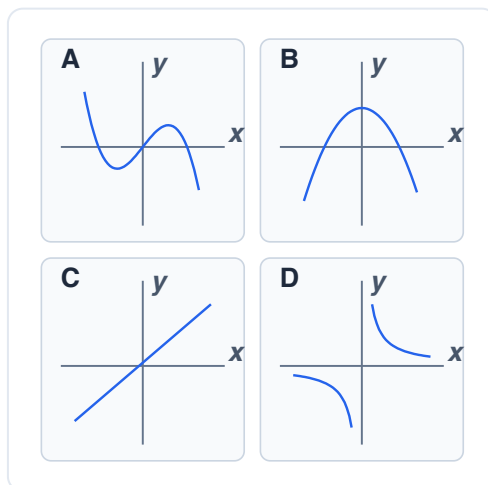
STEP-BY-STEP SOLUTION:

1. The equation $y = x^3 - 3x$ is a **positive cubic** ($+x^3$).
2. A positive cubic is an S-shaped curve starting in the bottom-left and finishing in the top-right.
3. This matches graph **B**.

B

EXAMPLE 3.2 **Non-Calculator**

Four sketched graphs A , B , C , and D are shown below.



Write down the letter of the graph that represents $y = \frac{4}{x}$.

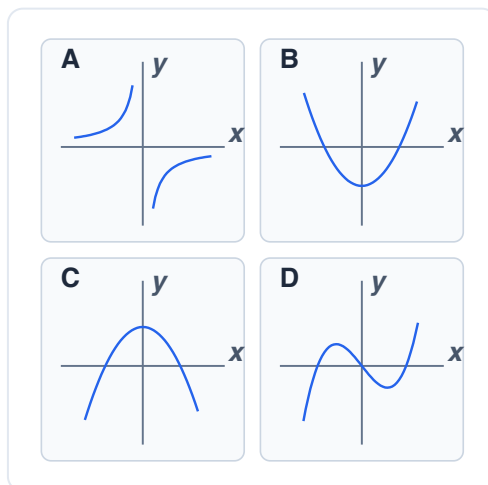
STEP-BY-STEP SOLUTION:

1. The equation $y = \frac{4}{x}$ is a **positive reciprocal** function.
2. Its graph consists of two smooth hyperbola branches in the top-right (Quadrant 1) and bottom-left (Quadrant 3) with asymptotes along the axes.
3. This matches graph **D**.

D

EXAMPLE 3.3 **Non-Calculator**

Four sketched graphs A , B , C , and D are shown below.



Write down the letter of the graph that represents $y = 4 - x^2$.

STEP-BY-STEP SOLUTION:

1. The equation $y = 4 - x^2$ is a **negative quadratic** because the x^2 term has a negative coefficient ($-x^2$).
2. A negative quadratic forms an inverted $\cap$ -shaped parabola with a maximum turning point at $(0, 4)$.
3. This matches graph **C**.

C

Ratio Contexts

1 Ratios in Geometry

□ CORE CONCEPT & METHOD

Ratios often appear in geometry problems, like splitting a straight line.

Example

ABC is a straight line. The length of BC is 3 times the length of AB . $AC = 120\text{m}$. Find BC .

If BC is 3 times AB , the ratio $AB : BC = 1 : 3$.

Total parts = $1 + 3 = 4$.

1 part = $120 \div 4 = 30\text{m}$.

BC is 3 parts, so $BC = 3 \times 30 = 90\text{m}$.

Common Mistake: Check whether the given number represents the TOTAL, ONE PART, or the DIFFERENCE between parts before dividing to find the value of 1 part.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

🚫 Non-Calculator

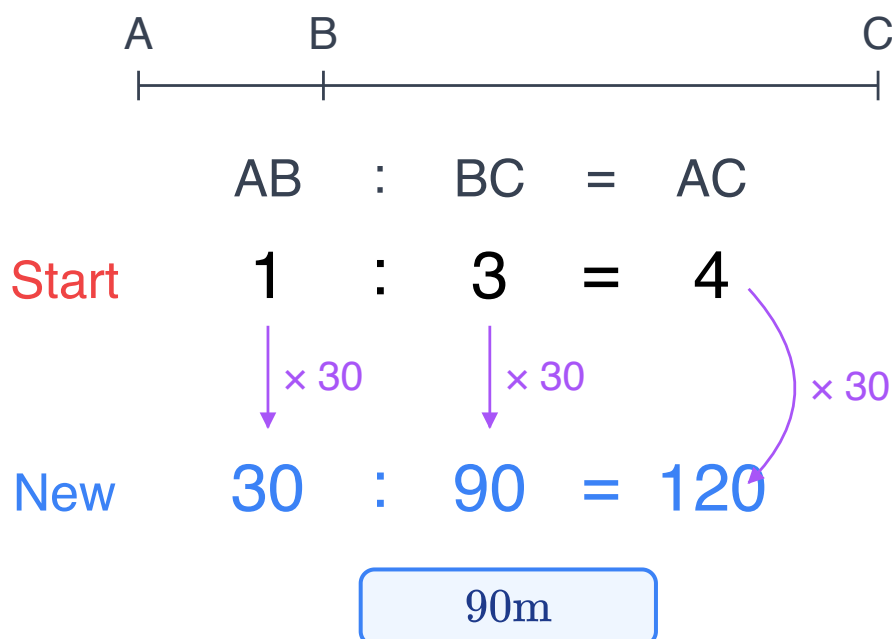
ABC is a straight line. The length of BC is 3 times the length of AB .

$AC = 120$ metres.

Work out the length of BC .

STEP-BY-STEP SOLUTION:

1. Ratio $AB : BC = 1 : 3$
2. Total parts = $1 + 3 = 4$
3. 1 part = $120 \div 4 = 30\text{m}$
4. $BC = 3 \times 30 = 90\text{m}$



2 Ratios and Fractions

□ CORE CONCEPT & METHOD

Ratios can be converted to fractions of the total. If $A : B = 3 : 2$, A has $\frac{3}{5}$ of the total.

Common Mistake: Work out each person's actual share first before taking fractions of their shares, and simplify the final fraction of the total fully.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

✎ Non-Calculator

Omar and Tariq share £180 in the ratio 4 : 5.

Omar then gives $\frac{1}{4}$ of his share to Zayd.

Tariq gives $\frac{1}{5}$ of his share to Zayd.

What fraction of the £180 does Zayd receive in total?

STEP-BY-STEP SOLUTION:

1. Share £180 in the ratio 4 : 5 (9 parts):

1 part = $£180 \div 9 = £20$

Omar = $4 \times £20 = £80$, Tariq = $5 \times £20 = £100$.

2. Zayd gets $\frac{1}{4}$ of $£80 = £20$ from Omar, and $\frac{1}{5}$ of $£100 = £20$ from Tariq.

3. Total for Zayd = $£20 + £20 = £40$.

4. Fraction of £180 = $\frac{40}{180} = \frac{2}{9}$.

3 Complex Contexts

□ CORE CONCEPT & METHOD

You might need to combine a ratio with a multiplier.

Common Mistake: Express all three quantities in a single consistent ratio (e.g. if Max gets 3 times Sam's 7 parts, Max has 21 parts, making 6 : 7 : 21) before summing total parts.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 Calculator

Dan, Sam and Max share £1,400.

The ratio of the amount Dan gets to the amount Sam gets is 5 : 6.

Max gets 1.5 times the amount Sam gets.

Work out the amount of money that Sam gets.

STEP-BY-STEP SOLUTION:

1. Find the parts for Max by multiplying Sam's parts by 1.5.
2. Add up all the parts to find the total.
3. Calculate the multiplier by dividing the total money by the total parts.

	Dan	:	Sam	$\times 1.5$	Max	=	Total
Start	5	:	6	:	9	=	20
	$\times 70$ ↓		$\times 70$ ↓		$\times 70$ ↓		$\times 70$ ↓
New	350	:	420	:	630	=	1400

£420

4 Changing Ratios

CORE CONCEPT & METHOD

When a quantity is transferred, the total amount stays the same, but the ratio changes. Use the total to find the value of one part before and after.

Common Mistake: When items are transferred from one person to another, the OVERALL TOTAL stays constant — find each person's count before and after using that unchanged total.

MODEL WORKED EXAMPLES

EXAMPLE 4.1

 **Calculator**

Leo and Toby have a total of 1980 trading cards.
The ratio of Leo's cards to Toby's cards is 7 : 5.

Leo gives Toby some trading cards.
The ratio of Leo's to Toby's cards is now 5 : 6.

Work out how many trading cards Leo gives to Toby.

STEP-BY-STEP SOLUTION:

- Before:** Find the total parts and the value of one part to determine how many cards Leo started with.
- After:** Use the same total (1980) but with the new ratio parts to find out how many cards Leo has now.
- Subtract the new amount from the old amount.

Before				After (Same Total)			
L	:	T	= Total	L	:	T	= Total
7	:	5	= 12	5	:	6	= 11
$\times 165$	$\downarrow$	$\times 165$	$\times 165$	$\times 180$	$\downarrow$	$\times 180$	$\times 180$
1155	:	825	= 1980	900	:	1080	= 1980



 - 255 cards

255

Ratios & Fractions

1 Ratio to Fraction

□ CORE CONCEPT & METHOD

To find what fraction a part is, put its ratio number over the **total** parts.

Example: Green and purple marbles are in the ratio 4:7.

Green = 4 parts.

Total parts = $4 + 7 = 11$.

Fraction that are green = $\frac{4}{11}$.

Common Mistake: A ratio of 3 : 5 means $3 + 5 = 8$ total parts, so the fractions are $\frac{3}{8}$ and $\frac{5}{8}$ — never write $\frac{3}{5}$ as the fraction of the total!

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✎ Non-Calculator

The ratio of Fiction to Non-Fiction books on a shelf is 5 : 4. What fraction of the books are Fiction?

STEP-BY-STEP SOLUTION:

Total parts = $5 + 4 = 9$.

The Fiction books are 5 parts out of the 9 total parts.

Fraction = $\frac{5}{9}$.

2 Fraction to Ratio

□ CORE CONCEPT & METHOD

If a fraction tells you how many are a certain type, the rest must make up the whole.

Example: $\frac{3}{8}$ of counters are silver. The rest are gold.

Silver = 3 parts.

Total = 8 parts.

Gold = $8 - 3 = 5$ parts.

Ratio of silver to gold = 3 : 5.

Common Mistake: If $\frac{2}{7}$ of a group are Type A, the remaining $7 - 2 = 5$ sevenths are Type B, so the ratio $A : B$ is 2 : 5 — never 2 : 7.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 **Non-Calculator**

$\frac{2}{7}$ of the trees in an orchard are apple trees and the rest are pear trees. What is the ratio of apple trees to pear trees?

STEP-BY-STEP SOLUTION:

If 2 parts out of 7 are apple trees, then $7 - 2 = 5$ parts are pear trees.

Ratio of apple trees to pear trees = 2 : 5.

3 Nested Ratios

□ CORE CONCEPT & METHOD

Sometimes a group is split by a ratio, and then a subgroup is split by another ratio.

Example:

Ratio of adults to children is 3:1. Ratio of boys to girls is 3:2.

Method 1: Multiplying Fractions

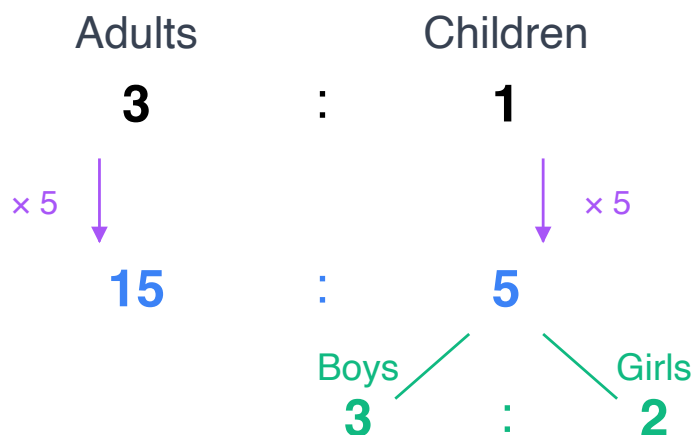
Fraction of children = $\frac{1}{4}$.

Fraction of children that are girls = $\frac{2}{5}$.

$$\text{Fraction of all people that are girls} = \frac{1}{4} \times \frac{2}{5} = \frac{2}{20} = \frac{1}{10}.$$

Method 2: Unifying the Ratios

Scale up the first ratio so its **Children** parts equal the total parts of the second ratio ($3 + 2 = 5$).



Now the ratios are unified: Adults : Boys : Girls = 15 : 3 : 2.

Total parts = $15 + 3 + 2 = 20$

$$\text{Fraction of girls} = \frac{2}{20} = \frac{1}{10}.$$

Common Mistake: Multiply the remaining fraction of the whole by the sub-group's fraction (e.g. $\frac{3}{4}$ left split in ratio 2 : 1 gives $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$), rather than adding denominators.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 Non-Calculator

A crate contains apples, oranges, and pears. $\frac{1}{4}$ of the fruit are apples. The ratio of oranges to pears is 2 : 1. What fraction of the fruit in the crate are pears?

STEP-BY-STEP SOLUTION:

If $\frac{1}{4}$ are apples, then $\frac{3}{4}$ are oranges and pears.

The oranges and pears are split in the ratio 2 : 1 (3 parts).

Pears are $\frac{1}{3}$ of the remaining fruit.

$$\text{Pear fraction} = \frac{1}{3} \text{ of } \frac{3}{4} = \frac{1}{3} \times \frac{3}{4} = \frac{3}{12} = \frac{1}{4}.$$

4 Ratios & Percentages

CORE CONCEPT & METHOD

Convert the percentage to a fraction and multiply it by the ratio fraction.

Example:

Men to women is 2:3. 20% of women wear glasses.

What fraction of everyone are women who wear glasses?

Method 1: Multiplying Fractions

$$\text{Women fraction} = \frac{3}{5}.$$

$$20\% = \frac{1}{5}.$$

$$\text{Fraction} = \frac{3}{5} \times \frac{1}{5} = \frac{3}{25}.$$

Method 2: Visualizing the Tree

A tree diagram shows us that the 20% applies exclusively to the Women's branch.

Men (M)

2

:

Women (W)

3

Glasses (G)

20%

No Glasses

80%

The Women's branch is $\frac{3}{5}$ of the total (2+3).

We want 20% ($\frac{1}{5}$) of that branch.

Fraction = $\frac{1}{5}$ of $\frac{3}{5}$ = $\frac{3}{25}$.

Common Mistake: For weighted percentages across ratio groups, multiply each group's fraction of the total by its percentage (never just average the two percentages!).

MODEL WORKED EXAMPLES

EXAMPLE 4.1

 **Non-Calculator**

In an animal shelter, the ratio of rabbits to guinea pigs is 3 : 7. What percentage of the animals are rabbits?

STEP-BY-STEP SOLUTION:

Total parts = 10. Rabbits are $\frac{3}{10}$ of the total.

To convert to a percentage, multiply by 100:

$$\frac{3}{10} \times 100 = 30\%.$$

Combining Ratios

1 Combining Ratios

□ CORE CONCEPT & METHOD

To combine $A : B$ and $B : C$, find a common multiple for the B parts so they match.

Example: $a : b = 3 : 4$ and $b : c = 5 : 2$

The common letter is b . In the first ratio, $b = 4$. In the second, $b = 5$.

The Lowest Common Multiple of 4 and 5 is 20.

Multiply $a : b$ by 5 $\rightarrow 15 : 20$

Multiply $b : c$ by 4 $\rightarrow 20 : 8$

Combined ratio $a : b : c = 15 : 20 : 8$

a		b		c	
3	:	4	:	□	$\rightarrow \times 5$
□	:	5	:	2	$\rightarrow \times 4$
↓					
15	:	20	:	□	
□	:	20	:	8	
—					
15	:	20	:	8	

Common Mistake: Identify the COMMON item appearing in both ratios and scale both ratios up to the Lowest Common Multiple (LCM) of that shared part.

EXAMPLE 1.1

 Non-Calculator

STEP-BY-STEP SOLUTION:

$$A : B : C = 4 : 5 : 7.$$

A		B		C
4	:	5	:	<input type="text"/>
<input type="text"/>	:	5	:	7
↓		↓		↓
<hr/>				
4	:	5	:	7

EXAMPLE 1.2

 **Non-Calculator**

STEP-BY-STEP SOLUTION:

Combined ratio: $A : B : C = 3 : 9 : 7$.

A	:	B	:	C	
1	:	3	:	<input type="text"/>	→ × 3
3	:	<input type="text"/>	:	7	
3	:	9	:	<input type="text"/>	
3	:	<input type="text"/>	:	7	
<hr/>					
3	:	9	:	7	

2 Overlapping Ratios (Lines & Sets)

BOOK CORE CONCEPT & METHOD

If you have ratios covering the same total length (e.g. AD), make the total parts the same!

Example: $AB : BD = 2 : 5$ and $AC : CD = 4 : 7$

Total for first ratio = 7 parts.

Total for second ratio = 11 parts.

Make total parts = 77 (LCM of 7 and 11).

First ratio $\times 11 \rightarrow AB = 22, BD = 55$.

Second ratio $\times 7 \rightarrow AC = 28, CD = 49$.

$BC = AC - AB = 28 - 22 = 6$.

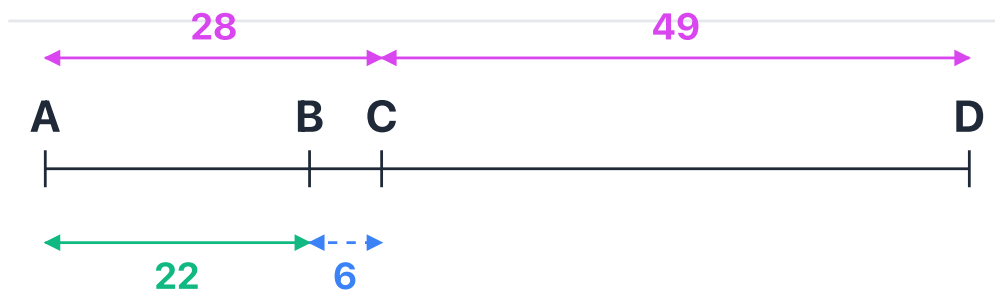
$AB : BC : CD = 22 : 6 : 49$.

AC : CD = AD (Total)

$$4 : 7 = 11 \xrightarrow{\times 7} 28 : 49 = 77$$

AB : BD = AD (Total)

$$2 : 5 = 7 \xrightarrow{\times 11} 22 : 55 = 77$$



Common Mistake: Check the order of the words carefully (e.g. if given "motorbikes to cars is 9 : 3", cars is the **SECOND** number 3, not 9) before matching the shared category.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

Non-Calculator

Points P, Q, and R lie on a line. $PQ : QR = 4 : 3$. What is the ratio $PQ : PR$?

STEP-BY-STEP SOLUTION:

Let $PQ = 4$ units and $QR = 3$ units. The whole line $PR = 4 + 3 = 7$ units.

The ratio $PQ : PR$ is $4 : 7$.

3 Combining by Totals

CORE CONCEPT & METHOD

If two groups share the **same** amount, their total ratio parts must be made equal.

Common Mistake: Once combined into a three-part ratio $x : y : z$, divide by the sum of ALL THREE parts ($x + y + z$) when finding a fraction of the whole.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 **Non-Calculator**

In a college, Boys : Girls = 2 : 3. 15% of the Boys study Physics. What fraction of the whole college are boys who study Physics?

STEP-BY-STEP SOLUTION:

Let there be 200 boys and 300 girls (total 500).

15% of 200 boys = 30 boys study Physics.

$$\text{Fraction} = \frac{30}{500} = \frac{3}{50}.$$

Linear Direct Proportion

□ CORE CONCEPT & METHOD

If y is directly proportional to x , we write $y \propto x$. This means that $y = kx$ for some constant k .

Step 1: Substitute the given values of y and x to find k .

Step 2: Write the full equation $y = kx$.

Step 3: Use the equation to find missing values.

Common Mistake: Direct proportion means $y = kx$ (passing through $(0, 0)$): divide y by x to find the constant multiplier k before finding missing values.

MODEL WORKED EXAMPLES

EXAMPLE 1

✎ Non-Calculator

y is directly proportional to x . When $x = 5$, $y = 35$. Find the value of y when $x = 9$.

STEP-BY-STEP SOLUTION:

Step 1: Start with $y = kx$. Substitute $x = 5$ and $y = 35$.

$$35 = k(5) \implies k = 7$$

Step 2: The equation is $y = 7x$.

Step 3: Substitute $x = 9$.

$$y = 7(9) = 63$$

Direct & Inverse Prop

1 Direct Proportion (Unitary)

📖 CORE CONCEPT & METHOD

In direct proportion, divide to find the value for ONE unit, then multiply for the required amount.

Example: 4 apples cost £1.20. How much for 9?

4 apples	$\div 4 \rightarrow$	1 apple	$\times 9 \rightarrow$	9 apples
£1.20	$\div 4 \rightarrow$	£0.30	$\times 9 \rightarrow$	£2.70

Common Mistake: For inverse proportion (workers vs time), MORE workers take LESS time — multiply (workers $\times$ days) to find the total worker-days needed for 1 worker first!

MODEL WORKED EXAMPLES

EXAMPLE 1.1

 Non-Calculator

6 identical notebooks cost £24. How much do 11 notebooks cost?

STEP-BY-STEP SOLUTION:

Find the cost of 1 notebook first (unitary method):

6 notebooks	$\div 6 \rightarrow$	1 notebook	$\times 11 \rightarrow$	11 notebooks
£24	$\div 6 \rightarrow$	£4	$\times 11 \rightarrow$	£44

2 Inverse Proportion (Unitary)

📖 CORE CONCEPT & METHOD

With inverse proportion (like workers completing a job), more workers means LESS time. Multiply to find the total time for ONE worker, then divide by the new number of workers.

Example: 4 gardeners take 5 days. How long for 2 gardeners?

4 gardeners	$\div 4 \rightarrow$	1 gardener	$\times 2 \rightarrow$	2 gardeners
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5 days

 $\times 4 \rightarrow$

20 days

 $\div 2 \rightarrow$

10 days

Note:

- $\div$ number of workers to find **1 worker**
- $\times$ number of days to find **days for 1 worker**
- **Inverse proportion — so do the opposite!**

Common Mistake: Read carefully whether the question asks for the TOTAL workers/pumps needed or how many MORE must be added to the existing team.

MODEL WORKED EXAMPLES**EXAMPLE 2.1** **Non-Calculator**

It takes 5 tilers 6 days to tile a hall. How long would it take 3 tilers?

STEP-BY-STEP SOLUTION:

More tilers = less time, so it is inverse proportion.

5 tilers

 $\div 5 \rightarrow$

1 tiler

 $\times 3 \rightarrow$

3 tilers

6 days

 $\times 5 \rightarrow$

30 days

 $\div 3 \rightarrow$

10 days

3 Using Proportion Formulas **CORE CONCEPT & METHOD**

If you are given a formula, simply substitute the known value and solve for the unknown.

Common Mistake: In algebraic inverse proportion ($y \propto \frac{1}{x}$), write $y = \frac{k}{x}$ and MULTIPLY $k = x \times y$ to find the constant — never divide $y \div x$.

MODEL WORKED EXAMPLES

EXAMPLE 3.1 **Non-Calculator**

F is inversely proportional to d . When $d = 8$, $F = 15$.

(a) Find a formula for F in terms of d .

(b) Work out the value of F when $d = 12$.

STEP-BY-STEP SOLUTION:

1. Inverse proportion means $F = \frac{k}{d}$, so $k = F \times d$.

2. Substitute $d = 8$ and $F = 15$: $k = 15 \times 8 = 120$, giving the formula **(a)** $F = \frac{120}{d}$.

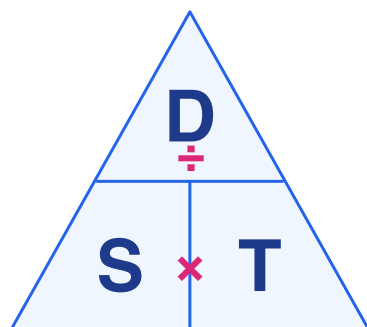
3. Substitute $d = 12$: **(b)** $F = \frac{120}{12} = 10$.

Speed, Distance, Time

1 Basic Speed, Distance, Time

□ CORE CONCEPT & METHOD

Remember the DST triangle! Cover up the one you want to find.



$$\text{Distance} = \text{Speed} \times \text{Time}$$

$$\text{Speed} = \text{Distance} \div \text{Time}$$

$$\text{Time} = \text{Distance} \div \text{Speed}$$

Note — To change:

- Minutes to hours: $\div 60$
- Seconds to minutes: $\div 60$

Careful with time: 4 hours 15 mins is 4.25 hours, NOT 4.15!

Common Mistake: Convert minutes to decimal hours by dividing by 60 (15 mins = 0.25 h, 45 mins = 0.75 h — never write 2 hours 15 mins as 2.15 hours!), and use Total Distance $\div$ Total Time for average speed.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Calculator

A car travels 180 km in 2.5 hours. Calculate the average speed of the car.

STEP-BY-STEP SOLUTION:

Use the formula:

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{Speed} = \frac{180}{2.5} = 72 \text{ km/h}$$

EXAMPLE 1.2 **Non-Calculator**

A cyclist travels at 15 km/h for 3.5 hours. How far does the cyclist travel?

STEP-BY-STEP SOLUTION:

Use the formula:

$$\begin{aligned}\text{Distance} &= \text{Speed} \times \text{Time} \\ \text{Distance} &= 15 \times 3.5 = 52.5 \text{ km}\end{aligned}$$

EXAMPLE 1.3 **Calculator**

A train travels 315 km at an average speed of 90 km/h. How long does the journey take?

STEP-BY-STEP SOLUTION:

Use the formula:

$$\begin{aligned}\text{Time} &= \frac{\text{Distance}}{\text{Speed}} \\ \text{Time} &= \frac{315}{90} = 3.5 \text{ hours}\end{aligned}$$

2 Journeys & Arrival Times **CORE CONCEPT & METHOD**

First calculate the time taken in hours. Convert any decimals to minutes by multiplying by 60. Then add it to the start time.

Common Mistake: When converting decimal hours back into hours and minutes for clock times, multiply the decimal part by 60 (e.g. 1.75 hours = 1 hour 45 mins, NOT 1 hour 75 mins).

MODEL WORKED EXAMPLES**EXAMPLE 2.1** **Non-Calculator**

A coach travels 210 km at a speed of 60 km/h. It departs at 09:40. What time does it arrive?

STEP-BY-STEP SOLUTION:

Step 1: Find the time taken in hours.

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}} = \frac{210}{60} = 3.5 \text{ hours}$$

Step 2: Convert the decimal to minutes.

$$0.5 \times 60 = 30 \text{ minutes}$$

Step 3: Add to departure time. 09:40 + 3 h 30 min = **13:10**

3 Average Speed for a Journey

CORE CONCEPT & METHOD

Never just average the two speeds! You must find the **Total Distance** and divide by the **Total Time**.

Common Mistake: Never average the speeds of two journey stages — always calculate

$$\text{Average Speed} = \frac{\text{Total Distance}}{\text{Total Time}} \text{ across the entire journey.}$$

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 **Calculator**

A driver travels 150 km at 75 km/h, then 180 km at 60 km/h. Find the average speed for the whole journey.

STEP-BY-STEP SOLUTION:

Step 1: Find time for each leg.

$$t_1 = \frac{150}{75} = 2 \text{ h}, \quad t_2 = \frac{180}{60} = 3 \text{ h}$$

Step 2: Find total distance and total time.

$$\begin{aligned} \text{Total Distance} &= 150 + 180 = 330 \text{ km} \\ \text{Total Time} &= 2 + 3 = 5 \text{ h} \end{aligned}$$

Step 3: Calculate average speed.

$$\text{Average Speed} = \frac{330}{5} = 66 \text{ km/h}$$

Common Mistake: If you incorrectly just found the average of the two speeds 75 and 60, you would get 67.5 km/h. This is why you must ALWAYS use Total Distance ÷ Total Time!

EXAMPLE 3.2

 **Calculator**

Tariq travels from Bristol to Birmingham at an average speed of 56 mph. The journey takes him 2 hours and 15 minutes.

Kofi makes the same journey in 1 hour and 45 minutes.

Work out Kofi's average speed.

STEP-BY-STEP SOLUTION:

Step 1: Find the distance Tariq travelled using:

$$\begin{aligned} \text{Distance} &= \text{Speed} \times \text{Time} \\ 2 \text{ hours } 15 \text{ mins} &= 2.25 \text{ hours} \\ \text{Distance} &= 56 \times 2.25 = 126 \text{ miles} \end{aligned}$$

Step 2: Work out Kofi's average speed (1 hour 45 mins = 1.75 hours):

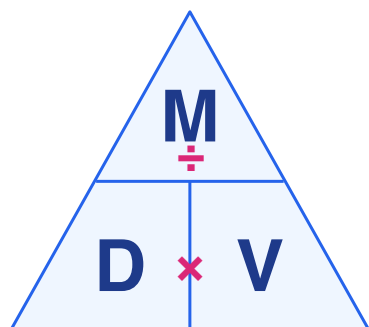
$$\begin{aligned} \text{Speed} &= \frac{\text{Distance}}{\text{Time}} \\ \text{Speed} &= \frac{126}{1.75} = 72 \text{ mph} \end{aligned}$$

Density, Mass, Volume

1 Basic Density, Mass, Volume

□ CORE CONCEPT & METHOD

Use the DMV triangle! Cover up the one you want to find.



$$\text{Mass} = \text{Density} \times \text{Volume}$$

$$\text{Density} = \text{Mass} \div \text{Volume}$$

$$\text{Volume} = \text{Mass} \div \text{Density}$$

Common Mistake: Check unit consistency (1 kg = 1,000 g) before using Density = Mass ÷ Volume, and for mixtures always use Total Mass ÷ Total Volume (never average the densities).

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Calculator

A block of copper has a mass of 356 g and a volume of 40 cm³. Calculate its density.

STEP-BY-STEP SOLUTION:

Use the formula:

$$\begin{aligned} \text{Density} &= \frac{\text{Mass}}{\text{Volume}} \\ \text{Density} &= \frac{356}{40} = 8.9 \text{ g/cm}^3 \end{aligned}$$

EXAMPLE 1.2

Non-Calculator

An oil has a density of 0.8 g/cm³. Find the volume of 360 g of the oil.

STEP-BY-STEP SOLUTION:

Rearrange for volume:

$$\begin{aligned} \text{Volume} &= \frac{\text{Mass}}{\text{Density}} \\ \text{Volume} &= \frac{360}{0.8} = 450 \text{ cm}^3 \end{aligned}$$

EXAMPLE 1.3 **Non-Calculator**

A timber beam has a volume of 60 cm^3 and a density of 0.7 g/cm^3 . Calculate its mass.

STEP-BY-STEP SOLUTION:

Use the formula:

$$\text{Mass} = \text{Density} \times \text{Volume}$$

$$\text{Mass} = 0.7 \times 60 = 42 \text{ g}$$

2 Mixtures of Liquids **CORE CONCEPT & METHOD**

To find the density of a mixture, you must find the **Total Mass** and divide it by the **Total Volume**. You cannot just average the densities!

Common Mistake: In mixture and alloy problems, find the mass and volume of each component separately so $\text{Total Mass} = m_A + m_B$ and $\text{Total Volume} = V_A + V_B$ before dividing.

MODEL WORKED EXAMPLES**EXAMPLE 2.1** **Calculator**

Syrup X has density 1.3 g/cm^3 and volume 150 cm^3 .

Syrup Y has density 0.9 g/cm^3 and volume 250 cm^3 .

The two syrups are mixed together.

Find the density of the mixture.

STEP-BY-STEP SOLUTION:

Step 1: Find the mass of each syrup using $\text{Mass} = \text{Density} \times \text{Volume}$.

$$m_X = 1.3 \times 150 = 195 \text{ g}, \quad m_Y = 0.9 \times 250 = 225 \text{ g}$$

Step 2: Find total mass and total volume.

$$\text{Total Mass} = 195 + 225 = 420 \text{ g}$$

$$\text{Total Volume} = 150 + 250 = 400 \text{ cm}^3$$

Step 3: Divide.

$$\text{Density} = \frac{420}{400} = 1.05 \text{ g/cm}^3$$

 **Calculator****EXAMPLE 2.2**

Solution P has a density of 1.5 g/cm^3 .

160 cm^3 of Solution P is mixed with Solution Q to make 240 cm^3 of Solution R.

Solution R has a density of 1.3 g/cm^3 .

Find the density of Solution Q.

STEP-BY-STEP SOLUTION:

Step 1: Find mass of Solution P:

$$\text{Mass}_P = 1.5 \times 160 = 240 \text{ g}$$

Step 2: Find mass of Solution R:

$$\text{Mass}_R = 1.3 \times 240 = 312 \text{ g}$$

Step 3: Find mass and volume of Solution Q:

$$\text{Mass}_Q = 312 - 240 = 72 \text{ g}$$

$$\text{Volume}_Q = 240 - 160 = 80 \text{ cm}^3$$

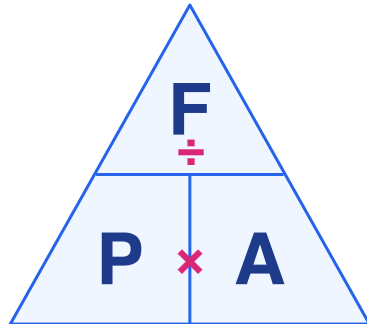
Step 4: Find density of Solution Q:

$$\text{Density}_Q = 72 \div 80 = 0.9 \text{ g/cm}^3$$

Pressure, Force, Area

CORE CONCEPT & METHOD

Use the PFA triangle! Cover up the one you want to find.



$$\text{Force} = \text{Pressure} \times \text{Area}$$

$$\text{Pressure} = \text{Force} \div \text{Area}$$

$$\text{Area} = \text{Force} \div \text{Pressure}$$

Common Mistake: When a 3D block rests on the ground, use the 2D AREA of the face touching the ground (not the volume or total surface area) in $\text{Pressure} = \text{Force} \div \text{Area}$ — and smallest contact area gives GREATEST pressure.

MODEL WORKED EXAMPLES

EXAMPLE 1

Non-Calculator

A crate exerts a force of 480 N on the floor.

The base of the crate has an area of 6 m².

Calculate the pressure on the floor.

STEP-BY-STEP SOLUTION:

Use the formula:

$$\begin{aligned} \text{Pressure} &= \frac{\text{Force}}{\text{Area}} \\ \text{Pressure} &= \frac{480}{6} = 80 \text{ N/m}^2 \end{aligned}$$

EXAMPLE 2

Calculator

A stone statue exerts a pressure of 45 N/m² and a force of 315 N.

Find the area of the base of the statue.

STEP-BY-STEP SOLUTION:

Rearrange for area:

$$\begin{aligned} \text{Area} &= \frac{\text{Force}}{\text{Pressure}} \\ \text{Area} &= \frac{315}{45} = 7 \text{ m}^2 \end{aligned}$$

EXAMPLE 3 **Non-Calculator**

A metal cylinder rests on a workbench. It exerts a pressure of 60 N/m^2 over an area of 0.4 m^2 .

Calculate the force exerted by the cylinder.

STEP-BY-STEP SOLUTION:

Use the formula:

$$\text{Force} = \text{Pressure} \times \text{Area}$$

$$\text{Force} = 60 \times 0.4 = 24 \text{ N}$$

GEOMETRY AND MEASURES — CIRCLES

Sector Area & Arc Length

1 Arc Length

CORE CONCEPT & METHOD

An arc is a fraction of the full circumference. The formula is:

$$\text{Arc Length} = \frac{\theta}{360} \times 2\pi r$$

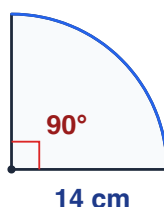
Common Mistake: For total PERIMETER of a sector, remember to add the two straight radii ($2r$) to the curved arc length: $\text{Perimeter} = \left(\frac{\theta}{360} \times 2\pi r \right) + 2r$.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

Find the arc length of this 90° sector with radius 14 cm in terms of π .



STEP-BY-STEP SOLUTION:

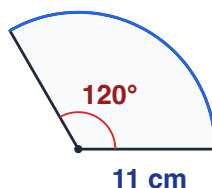
$$\begin{aligned} \text{Arc Length} &= \frac{90}{360} \times 2 \times \pi \times 14 \\ &= \frac{1}{4} \times 28\pi = 7\pi \text{ cm.} \end{aligned}$$

$$7\pi$$

EXAMPLE 1.2

Calculator

Calculate the arc length of this 120° sector with radius 11 cm. Give your answer to 1 decimal place.



STEP-BY-STEP SOLUTION:

$$\begin{aligned} \text{Arc Length} &= \frac{120}{360} \times 2 \times \pi \times 11 \\ &= \frac{1}{3} \times 22\pi \approx 23.04 = 23.0 \text{ cm.} \end{aligned}$$

$$23.0$$

2 Sector Area

□ CORE CONCEPT & METHOD

A sector is a fraction of the full area (like a slice of pizza). The formula is:

$$\text{Area} = \frac{\theta}{360} \times \pi r^2$$

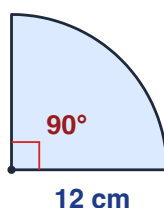
Common Mistake: Do not muddle the circle formulas: use πr^2 for sector AREA ($\frac{\theta}{360} \times \pi r^2$) and $2\pi r$ for ARC LENGTH ($\frac{\theta}{360} \times 2\pi r$).

MODEL WORKED EXAMPLES

EXAMPLE 2.1

☑ Non-Calculator

Find the area of this 90° sector with radius 12 cm in terms of π .



STEP-BY-STEP SOLUTION:

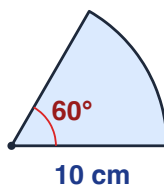
$$\begin{aligned} \text{Area} &= \frac{90}{360} \times \pi \times 12^2 \\ &= \frac{1}{4} \times 144\pi = 36\pi \text{ cm}^2. \end{aligned}$$

$$36\pi$$

EXAMPLE 2.2

☑ Calculator

Calculate the area of this 60° sector with radius 10 cm. Give your answer to 1 decimal place.



STEP-BY-STEP SOLUTION:

$$\begin{aligned} \text{Area} &= \frac{60}{360} \times \pi \times 10^2 \\ &= \frac{1}{6} \times 100\pi \approx 52.36 = 52.4 \text{ cm}^2. \end{aligned}$$

$$52.4$$

3 Sector Perimeter

□ CORE CONCEPT & METHOD

The perimeter of a sector includes the curved arc length PLUS the two straight radii:

$$\text{Perimeter} = \text{Arc Length} + 2r$$

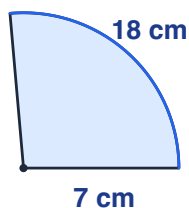
Common Mistake: If a question asks for an exact answer "in terms of π ", leave the π term and integer $2r$ separate (e.g. $6\pi + 18$) — never combine them into 24π !

MODEL WORKED EXAMPLES

EXAMPLE 3.1

🚫 Non-Calculator

A sector has an arc length of 18 cm and a radius of 7 cm. What is its total perimeter?



STEP-BY-STEP SOLUTION:

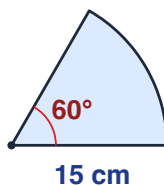
$$\begin{aligned} \text{Perimeter} &= \text{Arc} + \text{Radius} + \text{Radius} \\ &= 18 + 7 + 7 = 32 \text{ cm.} \end{aligned}$$

32

EXAMPLE 3.2

🧮 Calculator

Calculate the total perimeter of this 60° sector with radius 15 cm. Give your answer to 1 decimal place.



STEP-BY-STEP SOLUTION:

$$\text{Curved arc} = \frac{60}{360} \times 2\pi \times 15 = 5\pi \approx 15.71 \text{ cm.}$$

Add the two straight radii ($15 + 15 = 30$ cm):

$$\text{Perimeter} = 15.71 + 30 = 45.7 \text{ cm.}$$

45.7

4 Reverse Calculations

□ CORE CONCEPT & METHOD

If you are given the Area or Arc Length of a sector, substitute the known values into the formula and solve the equation backwards to find the missing angle θ or radius r !

Common Mistake: When working backwards from sector area $\frac{\theta}{360}\pi r^2 = k\pi$, cancel π from both sides first and remember to take the square root at the end if solving for r .

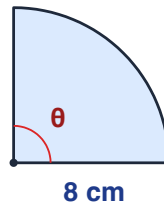
MODEL WORKED EXAMPLES

EXAMPLE 4.1

 **Non-Calculator**

A sector with radius 8 cm has an area of $16\pi \text{ cm}^2$. Find the angle θ .

Area = $16\pi \text{ cm}^2$



STEP-BY-STEP SOLUTION:

$$\frac{\theta}{360} \times \pi(8^2) = 16\pi$$

$$\frac{\theta}{360} \times 64 = 16$$

$$\frac{\theta}{360} = \frac{1}{4}$$

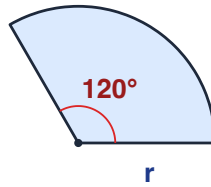
$$\theta = 90^\circ.$$

90°

EXAMPLE 4.2 **Non-Calculator**

A 120° sector has an area of $27\pi \text{ cm}^2$. Work out the radius r of the sector.

$$\text{Area} = 27\pi \text{ cm}^2$$

**STEP-BY-STEP SOLUTION:**

$$\frac{120}{360} \times \pi r^2 = 27\pi$$

$$\frac{1}{3} r^2 = 27 \implies r^2 = 81$$

$$r = \sqrt{81} = 9 \text{ cm.}$$

9

Enlargements

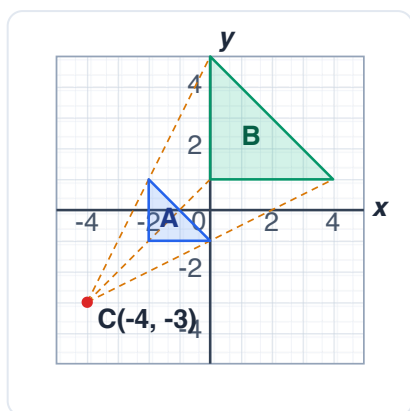
1 Enlargements: Positive Integer & Decimal Scale Factors

CORE CONCEPT & METHOD

Enlargements: Positive Integer & Decimal Scale Factors

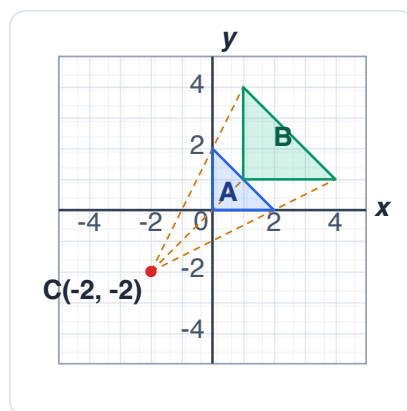
An **enlargement** scales the side lengths of a shape by a **scale factor** k and multiplies the distance of every vertex from the **centre of enlargement** $C(a, b)$ by k .

1. Enlarging from Centre $C(a, b)$ Using Vectors



- Count the vector $\vec{CP}$ from the centre C to a vertex P .
- Multiply both components by the scale factor k : $\vec{CP'} = k\vec{CP}$.
- Add $k\vec{CP}$ to the centre C : $P' = C + k(P - C)$.

2. Scale Factor & Ray Lines for the Centre



- Scale Factor:**

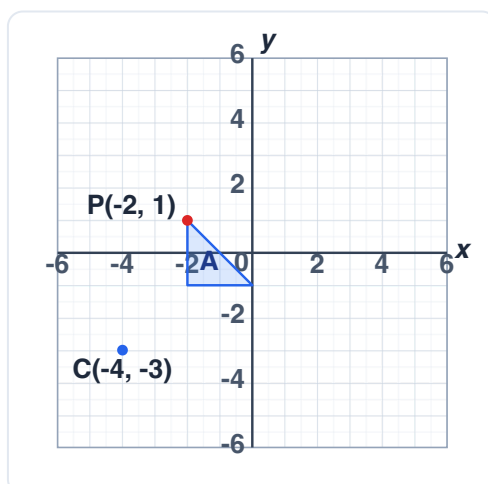
$$k = \frac{\text{Image side length (on B)}}{\text{Object side length (on A)}}$$
 (including decimals like 1.5 or 2.5).
- Finding C :** Draw straight ray lines through matching corners of B and A ; they cross at the centre C .

Common Mistake: Always count vectors from the CENTRE OF ENLARGEMENT C (not the origin), and use Scale Factor = Image Length $\div$ Object Length (so a smaller image has a fractional scale factor like $\frac{1}{2}$ or $\frac{1}{3}$).

MODEL WORKED EXAMPLES

EXAMPLE 1.1 **Non-Calculator**

Triangle A is drawn on the coordinate grid below with vertex $P(-2, 1)$.



Triangle A is enlarged by scale factor 2 about the centre of enlargement $C(-4, -3)$.
Find the coordinates of the image of vertex P .

STEP-BY-STEP SOLUTION:

Step 1: Find the vector from the centre $C(-4, -3)$ to vertex $P(-2, 1)$

$$\vec{CP} = \begin{pmatrix} -2 - (-4) \\ 1 - (-3) \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

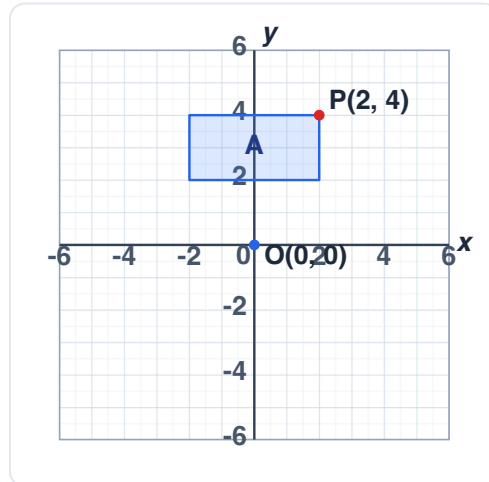
Step 2: Multiply the vector by the scale factor 2 and add to $C(-4, -3)$

$$\begin{aligned} \vec{CP'} &= 2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \\ P' &= (-4 + 4, -3 + 8) = (0, 5) \end{aligned}$$

$$(0, 5)$$

EXAMPLE 1.2 **Non-Calculator**

Rectangle A is drawn on the coordinate grid below with vertex $P(2, 4)$.



Rectangle A is enlarged by scale factor 1.5 with centre of enlargement at the origin $O(0, 0)$.
Find the coordinates of the image of vertex P .

STEP-BY-STEP SOLUTION:

Step 1: Multiply the coordinates from the origin $(0, 0)$ by 1.5

Since the centre of enlargement is $(0, 0)$, multiply both coordinates of $P(2, 4)$ by 1.5:

$$x' = 2 \times 1.5 = 3$$

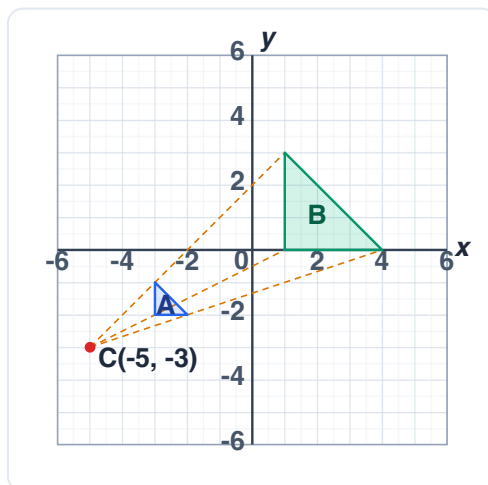
$$y' = 4 \times 1.5 = 6$$

The image of vertex P is $(3, 6)$.

$(3, 6)$

EXAMPLE 1.3 **Non-Calculator**

Triangle A is enlarged onto Triangle B , as shown on the coordinate grid below.



Find the scale factor and the coordinates of the centre of enlargement that maps Triangle A onto Triangle B .

STEP-BY-STEP SOLUTION:**Step 1: Compare matching side lengths to find the scale factor**

The base of Triangle A is 1 unit and the base of Triangle B is 3 units, so:

$$\text{Scale factor } k = \frac{3}{1} = 3$$

Step 2: Draw ray lines through corresponding vertices

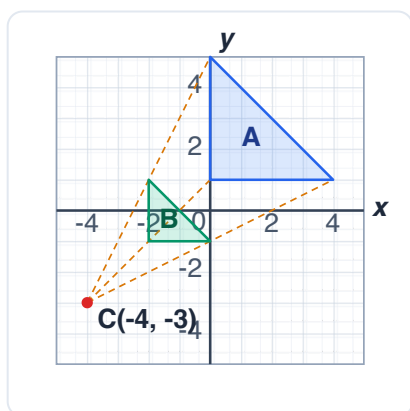
The ray line through $(1, 0)$ and $(-3, -2)$ and the ray line through $(1, 3)$ and $(-3, -1)$ intersect at $C(-5, -3)$.

$$k = 3, \quad C(-5, -3)$$

2 Fractional Scale Factors **CORE CONCEPT & METHOD****Enlargements with Fractional Scale Factors**

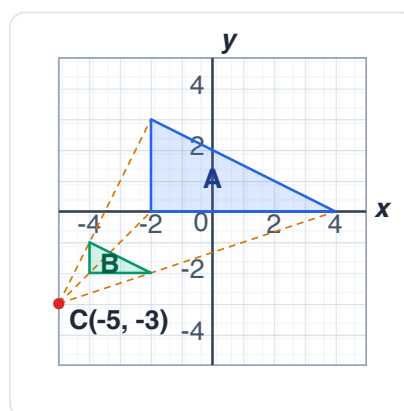
When the scale factor k is a fraction between 0 and 1 (such as $\frac{1}{2}$ or $\frac{1}{3}$), the transformation is still called an **enlargement** in GCSE Maths, even though the image is **smaller** than the object!

1. Enlarging by $k = \frac{1}{2}$ or $k = \frac{1}{3}$



- Multiply the vector $\vec{CP}$ from the centre C to each vertex P by the fraction k .
- Because $0 < k < 1$, the image B lies **between** the centre C and the object A !

2. Direction Matters: $A \rightarrow B$ vs $B \rightarrow A$



- Always check which shape is the **Image**:

$$k = \frac{\text{Image length}}{\text{Object length}}$$

- Here A has base 6 and B has base 2, so mapping A onto B has scale factor $\frac{2}{6} = \frac{1}{3}$ (not 3!).

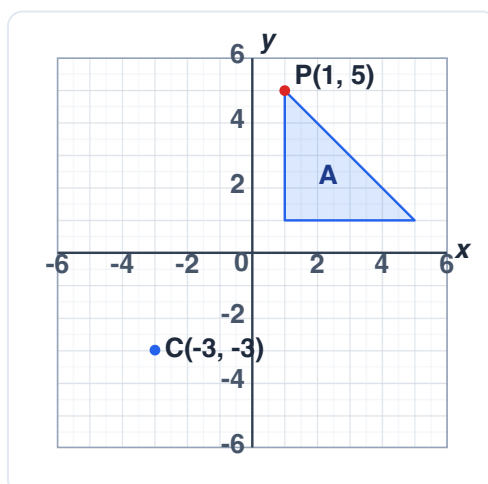
Common Mistake: When Shape A maps onto a smaller Shape B , the scale factor is a fraction less than 1 (e.g. $\frac{1}{2}$ or $\frac{1}{3}$, NEVER -2 or -3 — negative scale factors invert the shape).

MODEL WORKED EXAMPLES

EXAMPLE 2.1

Non-Calculator

Triangle A is drawn on the coordinate grid below with vertex $P(1, 5)$.



Triangle A is enlarged by scale factor $\frac{1}{2}$ about the centre $C(-3, -3)$.
Find the coordinates of the image of vertex P .

STEP-BY-STEP SOLUTION:

Step 1: Find the vector from $C(-3, -3)$ to $P(1, 5)$

$$\vec{CP} = \begin{pmatrix} 1 - (-3) \\ 5 - (-3) \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

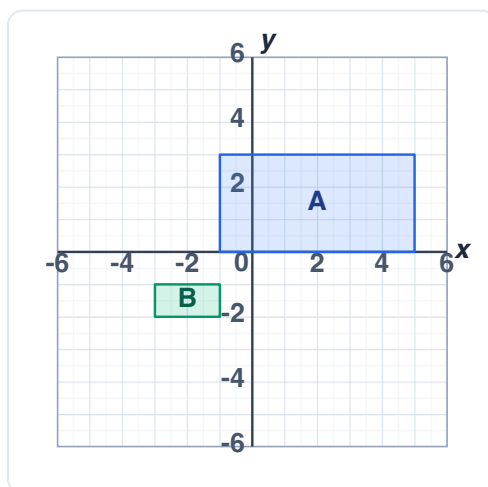
Step 2: Multiply by $\frac{1}{2}$ and add to $C(-3, -3)$

$$\begin{aligned} \vec{CP'} &= \frac{1}{2} \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ P' &= (-3 + 2, -3 + 4) = (-1, 1) \end{aligned}$$

$$(-1, 1)$$

EXAMPLE 2.2 **Non-Calculator**

Rectangle A is enlarged to give Rectangle B , as shown on the coordinate grid below.



Write down the scale factor of the enlargement that maps Rectangle A onto Rectangle B . Give your answer as a simplified fraction.

STEP-BY-STEP SOLUTION:

Step 1: Measure corresponding horizontal bases on A and B

The base of Rectangle A (object) is 6 units, and the base of Rectangle B (image) is 2 units.

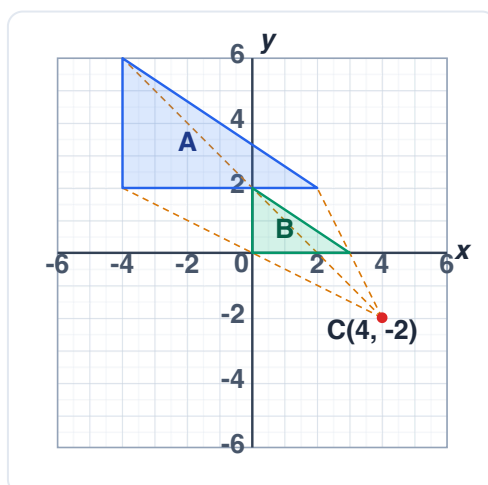
Step 2: Calculate $\frac{\text{Image length}}{\text{Object length}}$

$$\text{Scale factor} = \frac{2}{6} = \frac{1}{3}$$

$$\frac{1}{3}$$

EXAMPLE 2.3**Non-Calculator**

Triangle A is enlarged by scale factor $\frac{1}{2}$ onto Triangle B , as shown on the coordinate grid below.



Find the coordinates of the centre of enlargement.

STEP-BY-STEP SOLUTION:

Step 1: Draw ray lines from the vertices of A through the matching vertices of B

- The line through $(-4, 2)$ and $(0, 0)$ continues to $(4, -2)$.
- The line through $(2, 2)$ and $(3, 0)$ continues to $(4, -2)$.

Both rays intersect at the centre of enlargement $C(4, -2)$.

$(4, -2)$

Vectors

1 Column Vector Arithmetic

□ CORE CONCEPT & METHOD

A column vector like $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ means '2 right, 3 up'. To multiply by a scalar, multiply both numbers. To add them, add the tops and add the bottoms.

Common Mistake: Reversing the direction of a vector flips its sign ($\overrightarrow{PO} = -\overrightarrow{OP}$), and always follow a connected nose-to-tail path of known vectors.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

 Non-Calculator

Let $\mathbf{p} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Work out $4\mathbf{p} - \mathbf{q}$ as a column vector.

STEP-BY-STEP SOLUTION:

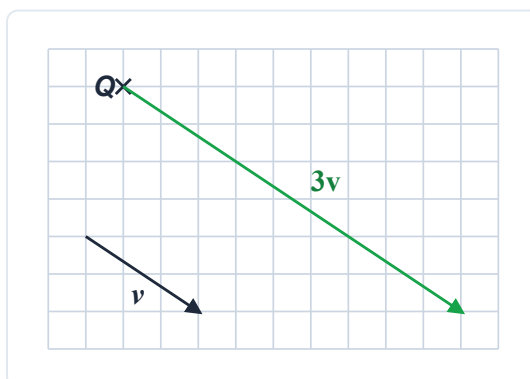
$$4\mathbf{p} = 4 \times \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 20 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} 20 \\ -8 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 17 \\ -12 \end{pmatrix}$$

$$\begin{pmatrix} 17 \\ -12 \end{pmatrix}$$

EXAMPLE 1.2 **Non-Calculator**

The vector $\mathbf{v}$ is drawn in black on the grid below. Work out the column vector for $\mathbf{v}$, and verify the green vector $3\mathbf{v}$ drawn starting from point Q .

**STEP-BY-STEP SOLUTION:**

1. Vector $\mathbf{v}$ travels 3 squares right and 2 squares down, so $\mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.
2. Multiply both components by 3: $3\mathbf{v} = 3 \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \end{pmatrix}$, so the green arrow $3\mathbf{v}$ from Q travels 9 units right and 6 units down.

2 Vectors from Coordinates **CORE CONCEPT & METHOD**

To find the displacement vector $\vec{AB}$ from point A to point B , subtract the coordinates of A from the coordinates of B .

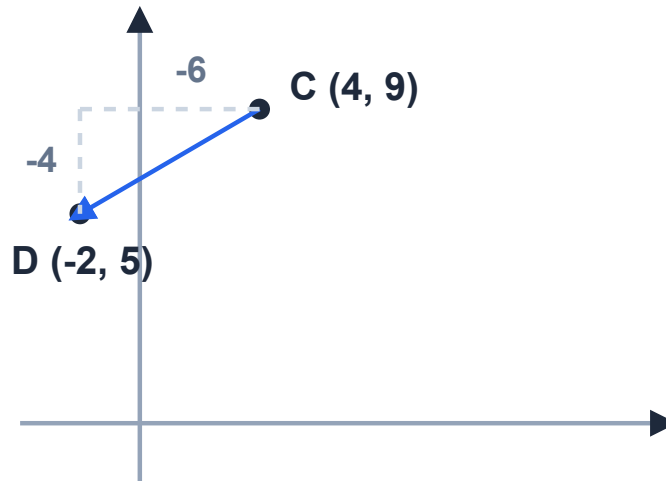
Common Mistake: For the column vector $\vec{AB}$ from $A(x_1, y_1)$ to $B(x_2, y_2)$, always compute $\text{End} - \text{Start} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$, never $\text{Start} - \text{End}$.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

Non-Calculator

C is the point $(4, 9)$ and D is the point $(-2, 5)$. Write down as a column vector $\vec{CD}$.



STEP-BY-STEP SOLUTION:

$$\text{x-change} = -2 - 4 = -6$$

$$\text{y-change} = 5 - 9 = -4$$

$$\vec{CD} = \begin{pmatrix} -6 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} -6 \\ -4 \end{pmatrix}$$

3 Triangle Geometry

CORE CONCEPT & METHOD

To travel from A to B , you can take a detour through O : $\vec{AB} = \vec{AO} + \vec{OB}$.

Remember that reversing a vector's direction makes it negative (e.g. $\vec{AO} = -\vec{OA}$).

Common Mistake: If P divides AB in the ratio $1 : 2$, there are $1 + 2 = 3$ total parts, so $\vec{AP} = \frac{1}{3}\vec{AB}$
 — never $\frac{1}{2}\vec{AB}$!

MODEL WORKED EXAMPLES

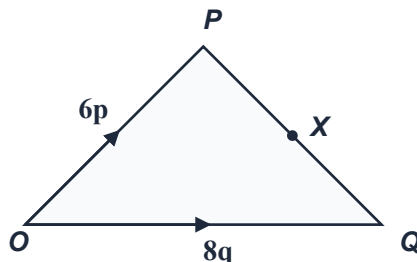
EXAMPLE 3.1

Non-Calculator

In triangle OPQ , $\vec{OP} = 6\mathbf{p}$ and $\vec{OQ} = 8\mathbf{q}$.

X is the midpoint of PQ .

Find, in terms of $\mathbf{p}$ and $\mathbf{q}$, the vector $\vec{OX}$.



STEP-BY-STEP SOLUTION:

1. Find $\vec{PQ}$:

$$\vec{PQ} = \vec{PO} + \vec{OQ} = -6\mathbf{p} + 8\mathbf{q}$$

2. Find $\vec{PX}$:

$$\text{Since } X \text{ is the midpoint, } \vec{PX} = \frac{1}{2}\vec{PQ} = -3\mathbf{p} + 4\mathbf{q}$$

3. Find $\vec{OX}$:

$$\vec{OX} = \vec{OP} + \vec{PX} = 6\mathbf{p} - 3\mathbf{p} + 4\mathbf{q} = 3\mathbf{p} + 4\mathbf{q}$$

$$\mathbf{3p + 4q}$$

4 Shape Geometry

CORE CONCEPT & METHOD

In regular hexagons and parallelograms, many sides are parallel and equal in length.

For example, in a hexagon with centre O , $\vec{AB}$ is parallel and equal to $\vec{ED}$, and $\vec{OA}$ is equal to $\vec{CB}$.

Common Mistake: In regular polygons and parallelograms, parallel sides of equal length have the SAME vector only when travelled in the SAME direction.

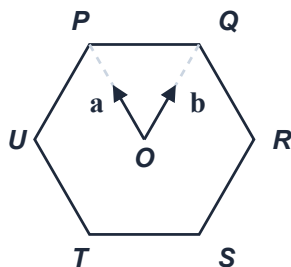
MODEL WORKED EXAMPLES

EXAMPLE 4.1 **Non-Calculator**

$PQRSTU$ is a regular hexagon with centre O .

$\vec{OP} = \mathbf{a}$ and $\vec{OQ} = \mathbf{b}$.

Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the vectors **(i)** $\vec{PQ}$ and **(ii)** $\vec{PS}$.

**STEP-BY-STEP SOLUTION:**

(i) Find $\vec{PQ}$:

Go via the centre O : $\vec{PQ} = \vec{PO} + \vec{OQ} = -\mathbf{a} + \mathbf{b}$

(ii) Find $\vec{PS}$:

The diagonal PS passes straight through the centre O , and $\vec{PO} = \vec{OS} = -\mathbf{a}$.

So, $\vec{PS} = 2\vec{PO} = -2\mathbf{a}$.

Spheres and Cones

1 Spheres and Cones Basics

□ CORE CONCEPT & METHOD

Sphere: Volume = $\frac{4}{3}\pi r^3$, Surface Area = $4\pi r^2$

Cone: Volume = $\frac{1}{3}\pi r^2 h$, Curved Surface Area = $\pi r l$

(Don't forget to add the base πr^2 if it's a *solid* cone!)

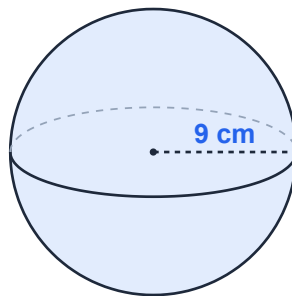
Common Mistake: Halve the diameter to get the radius r FIRST, and use Pythagoras ($l^2 = r^2 + h^2$) to find the slant height l for the curved surface area $\pi r l$ of a cone.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✎ Non-Calculator

Find the volume and surface area of a sphere with a radius of 9 cm. Give your answers in terms of π .



STEP-BY-STEP SOLUTION:

1. Volume

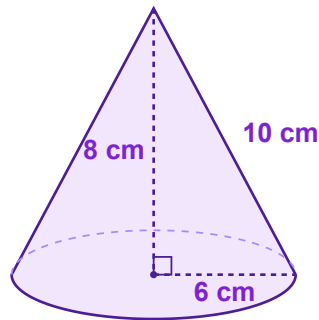
$$\begin{aligned}\text{Volume} &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \pi \times 9^3 \\ &= \frac{4}{3} \times 729 \times \pi = 972\pi \text{ cm}^3\end{aligned}$$

2. Surface Area

$$\begin{aligned}\text{SA} &= 4\pi r^2 \\ &= 4 \times \pi \times 9^2 \\ &= 4 \times 81 \times \pi = 324\pi \text{ cm}^2\end{aligned}$$

EXAMPLE 1.2**Non-Calculator**

A cone has a base radius of 6 cm, a perpendicular height of 8 cm, and a slant height of 10 cm. Find its volume and curved surface area in terms of π .

**STEP-BY-STEP SOLUTION:****1. Volume**

$$\begin{aligned}\text{Volume} &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3} \times \pi \times 6^2 \times 8 \\ &= \frac{1}{3} \times 36 \times 8 \times \pi = 96\pi \text{ cm}^3\end{aligned}$$

2. Curved Surface Area

$$\begin{aligned}\text{CSA} &= \pi r l \\ &= \pi \times 6 \times 10 = 60\pi \text{ cm}^2\end{aligned}$$

2 Composite Solids**□ CORE CONCEPT & METHOD**

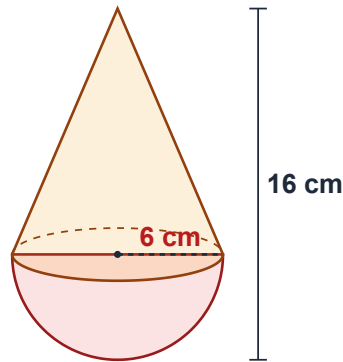
Split the shape into its standard parts (e.g., Cone + Hemisphere). Remember that a hemisphere's volume is exactly half of a full sphere: $\frac{2}{3}\pi r^3$.

Common Mistake: For the TOTAL surface area of a solid cone or solid hemisphere, include the flat circular base πr^2 — but omit the internal joined circle on a composite solid.

MODEL WORKED EXAMPLES

EXAMPLE 2.1**Non-Calculator**

A composite solid is made of a cone placed on top of a hemisphere. The radius is 6 cm and the total height of the solid is 16 cm. Find the total volume in terms of π .

**STEP-BY-STEP SOLUTION:****Step 1: Identify the heights.**

The radius of the hemisphere is 6 cm, which means it reaches 6 cm downwards.

The total height is 16 cm, so the cone's height is $16 - 6 = 10$ cm.

Step 2: Volume of Hemisphere.

$$V = \frac{2}{3}\pi r^3$$

$$= \frac{2}{3} \times \pi \times 6^3 = 144\pi \text{ cm}^3$$

Step 3: Volume of Cone.

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3} \times \pi \times 6^2 \times 10 = 120\pi \text{ cm}^3$$

Step 4: Total Volume.

$$144\pi + 120\pi = 264\pi \text{ cm}^3$$

3 Problem Solving

□ CORE CONCEPT & METHOD

If a sphere is dropped into a tank of water, it displaces its exact volume! Set the **Volume of Sphere** equal to the **Volume of the displaced water (Base Area × Rise)**.

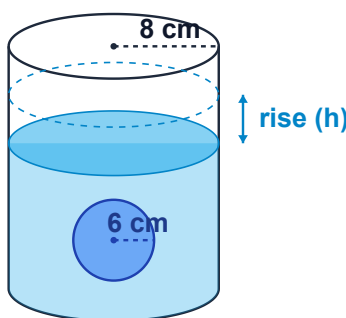
Common Mistake: When a solid is melted/recast or submerged in water, equate the 3D VOLUMES (e.g. Volume of submerged sphere = Tank base area × water rise h).

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 **Calculator**

A solid metal sphere of radius 6 cm is dropped into a cylindrical tank of water with radius 8 cm. The sphere is completely submerged. Calculate the rise in the water level.



STEP-BY-STEP SOLUTION:

Step 1: Volume of the Sphere

This is the volume of water displaced.

$$V_{\text{sphere}} = \frac{4}{3}\pi(6^3) = 288\pi$$

Step 2: Volume of the Rising Water

The displaced water forms a cylinder (base radius 8) with a height equal to the rise h .

$$V_{\text{water}} = \pi r^2 h = \pi(8^2)h = 64\pi h$$

Step 3: Equate and Solve for h

$$64\pi h = 288\pi$$

Cancel the π :

$$\begin{aligned} 64h &= 288 \\ h &= \frac{288}{64} = 4.5 \text{ cm} \end{aligned}$$

Congruent Triangles

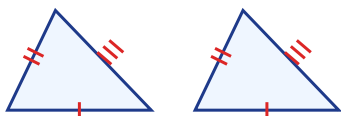
1 Cheat Sheet: Recognising Congruency

CORE CONCEPT & METHOD

Two triangles are **congruent** if they are identical in both shape and size (one can be rotated, reflected, or translated to fit exactly onto the other). To prove congruence, show **one** of these four conditions:

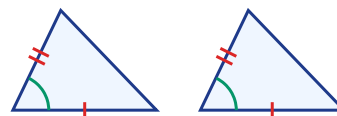
1. SSS (Side–Side–Side)

All three pairs of corresponding sides are equal in length.



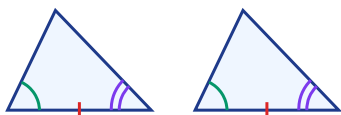
2. SAS (Side–Angle–Side)

Two sides and the **included angle** between them are equal.



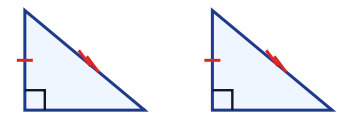
3. ASA (Angle–Side–Angle)

Two angles and a corresponding side are equal (includes AAS since angles sum to 180°).



4. RHS (Right angle–Hypotenuse–Side)

Both triangles have a 90° angle, equal hypotenuses, and one other equal side.



Common Mistake: AAA (three equal angles) proves SIMILARITY, never congruence — and in SAS, the equal angle **MUST** be the included angle sandwiched between the two equal sides.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

Triangles ABC and DEF have $AB = DE = 8$ cm, $BC = EF = 6$ cm, and $\angle ABC = \angle DEF = 55^\circ$. State the condition that proves $\triangle ABC$ and $\triangle DEF$ are congruent.

STEP-BY-STEP SOLUTION:

The equal 55° angle lies directly between the two equal pairs of sides (8 cm and 6 cm), so it is the included angle. Therefore the triangles are congruent by **SAS**.

2 Congruent Triangles

CORE CONCEPT & METHOD

To earn full marks on a 3-mark Edexcel congruence proof ("Prove that $\triangle ABC$ is congruent to $\triangle DEF$ "), set out **three justified statements** followed by a concluding condition:

1. State three equal pairs of sides/angles with a geometric reason in brackets on every line:

- Shared edge: " AC is a common side" | Midpoint/bisector: " $AM = CM$ (M is the midpoint of AC)"
- Parallel lines: " $\text{alternate angles are equal}$ " or " $\text{corresponding angles are equal}$ "

- Crossing lines / circles: "vertically opposite angles are equal" or "radii of the same circle are equal"

2. State the conclusion clearly: "Therefore $\triangle ABC \equiv \triangle DEF$ by **SSS / SAS / ASA / RHS**."

Common Mistake: In a 3-mark congruence proof, write a geometric reason beside EVERY equal pair (e.g. "common side", "alternate angles", "vertically opposite") and end with the exact 3-letter condition.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 **Non-Calculator**

$ABCD$ is a parallelogram with diagonal AC drawn. Prove that triangle ABC is congruent to triangle CDA .

STEP-BY-STEP SOLUTION:

1. $AB = CD$ (opposite sides of a parallelogram are equal).
 2. $\angle BAC = \angle DCA$ (alternate angles, since $AB \parallel DC$).
 3. AC is a common side to both triangles.
- Therefore $\triangle ABC \equiv \triangle CDA$ by **SAS**.

Trigonometry Exact Values

1 Recall Exact Values

□ CORE CONCEPT & METHOD

You must memorise the exact values of $\sin$, $\cos$, and $\tan$ for 0° , 30° , 45° , 60° , and 90° .

The Hand / Table Trick

Here is a brilliant trick to instantly recall any exact value:

	0°	30°	45°	60°	90°
Sin	0	1	2	3	4
Cos	4	3	2	1	0
	$\sqrt{\hspace{1.5cm}}$				
	$\underline{\hspace{1.5cm}}$				
	2				

- Write the angles across the top.
- For **Sin**, count up: **0, 1, 2, 3, 4**.
- For **Cos**, count down: **4, 3, 2, 1, 0**.
- Put a massive **Square Root** over all the numbers, and divide everything by **2**.

What about Tan?

Just divide the Sin value by the Cos value!

$$\tan = \frac{\sin}{\cos}$$

Common Mistake: Memorise $\sin 30^\circ = \frac{1}{2}$, $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\tan 30^\circ = \frac{1}{\sqrt{3}}$,

$\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}$, $\tan 45^\circ = 1$, $\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{1}{2}$, $\tan 60^\circ = \sqrt{3}$ for Non-Calculator Paper 1.

MODEL WORKED EXAMPLES

EXAMPLE 1.1 **Non-Calculator**Find the exact value of $\cos(30^\circ)$.**STEP-BY-STEP SOLUTION:**

Using the table trick:

1. Look at the **Cos** row.
2. The number under 30° is **3**.
3. Put a square root over it and divide by 2.

$$\cos(30^\circ) = \frac{\sqrt{3}}{2}$$

EXAMPLE 1.2 **Non-Calculator**Find the exact value of $\sin(45^\circ)$.**STEP-BY-STEP SOLUTION:**

Using the table trick:

1. Look at the **Sin** row.
2. The number under 45° is **2**.
3. Put a square root over it and divide by 2.

$$\sin(45^\circ) = \frac{\sqrt{2}}{2}$$

EXAMPLE 1.3 **Non-Calculator**Find the exact value of $\tan(60^\circ)$.**STEP-BY-STEP SOLUTION:**Remember that $\tan = \frac{\sin}{\cos}$.

1. For 60° , the Sin number is **3** and the Cos number is **1**.

$$2. \tan(60^\circ) = \frac{\sqrt{3}}{\sqrt{1}} = \sqrt{3}$$

EXAMPLE 1.4 **Non-Calculator**Evaluate $\cos(30^\circ) \times \tan(60^\circ)$.**STEP-BY-STEP SOLUTION:**

Using exact values:

$$\cos(30^\circ) = \frac{\sqrt{3}}{2}$$

$$\tan(60^\circ) = \frac{\sqrt{3}}{\sqrt{1}} = \sqrt{3}$$

Multiply them together:

$$\frac{\sqrt{3}}{2} \times \sqrt{3} = \frac{3}{2}$$

2 Using Exact Values

□ CORE CONCEPT & METHOD

Use SOH CAH TOA, substitute the exact fraction or surd for the trig ratio, and solve the equation to find the missing side.

Common Mistake: When dividing by a fraction like $\sin 60^\circ = \frac{\sqrt{3}}{2}$, multiply by the reciprocal $\frac{2}{\sqrt{3}}$ and rationalise the denominator fully (e.g. $\frac{24}{\sqrt{3}} = \frac{24\sqrt{3}}{3} = 8\sqrt{3}$).

MODEL WORKED EXAMPLES

EXAMPLE 2.1

✎ Non-Calculator

Find the exact length of the hypotenuse in a right-angled triangle where the adjacent side is 7 cm and the angle is 45° .

STEP-BY-STEP SOLUTION:

Using Cosine (CAH):

$$\cos(45^\circ) = \frac{\text{Adj}}{\text{Hyp}}$$

$$\frac{\sqrt{2}}{2} = \frac{7}{H}$$

Cross-multiply:

$$H\sqrt{2} = 14$$

$$H = \frac{14}{\sqrt{2}} = 7\sqrt{2} \text{ cm}$$

EXAMPLE 2.2

✎ Non-Calculator

A right-angled triangle has an angle of 60° and a hypotenuse of 16 cm. Find the exact length of the opposite side.

STEP-BY-STEP SOLUTION:

Using Sine (SOH):

$$\sin(60^\circ) = \frac{\text{Opp}}{\text{Hyp}}$$

$$\frac{\sqrt{3}}{2} = \frac{\text{Opp}}{16}$$

Multiply both sides by 16:

$$\text{Opp} = 16 \times \frac{\sqrt{3}}{2}$$

$$\text{Opp} = 8\sqrt{3} \text{ cm}$$

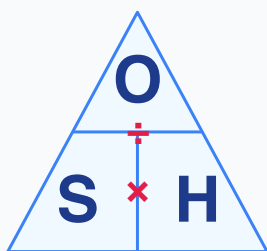
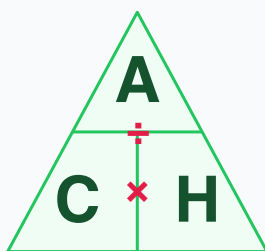
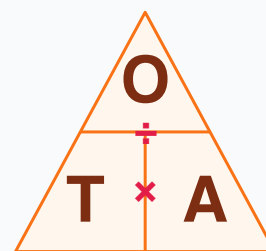
SOHCAHTOA

1 Introduction to Trig

□ CORE CONCEPT & METHOD

Trigonometry uses three special ratios (**Sine**, **Cosine**, and **Tangent**) to link the angles and sides of right-angled triangles.

Just like the Speed-Distance-Time triangle, you can use these **Magic Triangles** to instantly find any formula. To find a missing part, simply cover it up with your finger!



- If you want to find the **O**pposite, cover up O: you are left with $S \times H$.
- If you want to find the **H**ypotenuse, cover up H: you are left with $\frac{O}{S}$.
- If you want to find an angle (like **C**os), cover up C: you are left with $\frac{A}{H}$.

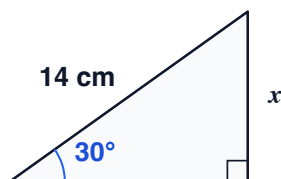
Common Mistake: Label Opposite (across from the angle) and Hypotenuse (across from the 90° box) first, and use $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ ONLY when finding an angle.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

 Calculator
Example 1: Finding a Missing Side

A right-angled triangle has an angle of 30° and a hypotenuse of 14 cm. Find the length of the opposite side.

**STEP-BY-STEP SOLUTION:**

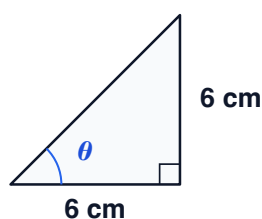
1. We know the **Hypotenuse (H)** and want the **Opposite (O)**.
2. This matches **SOH**.
3. To find O, cover up O in the magic triangle.
4. You are left with $S \times H$.

$$\text{Opposite} = \sin(30^\circ) \times 14 = 7 \text{ cm}$$

EXAMPLE 1.2

 Calculator
Example 2: Finding a Missing Angle

A right-angled triangle has an opposite side of 6 cm and an adjacent side of 6 cm. Find the missing angle θ .

**STEP-BY-STEP SOLUTION:**

1. We know the **Opposite (O)** and **Adjacent (A)**.
2. This matches **TOA**.
3. To find the angle (T), cover up T in the magic triangle.
4. You are left with $\frac{O}{A}$.

$$\tan(\theta) = \frac{6}{6} = 1$$

To find the angle, use the inverse $\tan^{-1}$ button:

$$\theta = \tan^{-1}(1) = 45^\circ$$

2 Finding Missing Sides

📖 CORE CONCEPT & METHOD

Remember **SOH CAH TOA**.

1. Label the sides: Hypotenuse, Opposite, Adjacent.
2. Choose the correct formula based on what you have and what you want to find.
3. Substitute the values and solve for x .

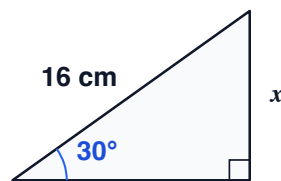
Common Mistake: When the unknown x is on the denominator (e.g. $\tan 23^\circ = \frac{24}{x}$), swap x and $\tan 23^\circ$ to divide ($x = \frac{24}{\tan 23^\circ}$) — never multiply $24 \times \tan 23^\circ$.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 **Calculator**

In a right-angled triangle, the hypotenuse is 16 cm and the angle is 30° . Find the length of the opposite side.



STEP-BY-STEP SOLUTION:

We know the Hypotenuse (H) and want the Opposite (O). This means we use Sine (SOH).

$$\begin{aligned}\sin(\theta) &= \frac{O}{H} \\ \sin(30^\circ) &= \frac{x}{16} \\ x &= 16 \times \sin(30^\circ) = 16 \times 0.5 = 8 \text{ cm}\end{aligned}$$

3 Finding Missing Angles

□ CORE CONCEPT & METHOD

To find an angle, you must use the **inverse trigonometric functions**: $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$.

Calculator Tip

Whenever you need to find an angle, look for the button with **to the power of -1** (e.g. $\sin^{-1}$). You usually get this by pressing the **SHIFT** or **2ND** button, and then the normal Sin, Cos, or Tan button.

Example: If $\sin(x) = 0.5$, you type $\sin^{-1}(0.5)$ into your calculator to find x .

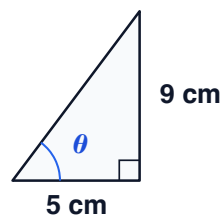
Common Mistake: Check your calculator is in DEGREES mode (D), and never round intermediate lengths early in multi-step right-angled triangle problems.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 **Calculator**

Find the size of the angle if the opposite side is 9 cm and the adjacent side is 5 cm.



STEP-BY-STEP SOLUTION:

We know the Opposite (O) and Adjacent (A), so we use Tangent (TOA).

$$\tan(\theta) = \frac{O}{A}$$

$$\tan(\theta) = \frac{9}{5}$$

$$\theta = \tan^{-1}(1.8) \approx 60.9^\circ$$

4 Problem Solving

□ CORE CONCEPT & METHOD

Break complex shapes down into right-angled triangles. You may need to use Pythagoras' Theorem alongside Trigonometry!

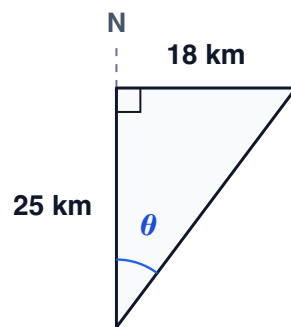
Common Mistake: In double-triangle and trapezium problems, split the shape with a perpendicular altitude and keep full calculator accuracy on the shared side until the final rounding step.

MODEL WORKED EXAMPLES

EXAMPLE 4.1

 **Calculator**

A ferry sails 25 km due North, then 18 km due East. What is the bearing of the ferry from its starting point?



STEP-BY-STEP SOLUTION:

Draw a right-angled triangle. The adjacent side (North) is 25, and the opposite side (East) is 18. The bearing is the angle clockwise from North.

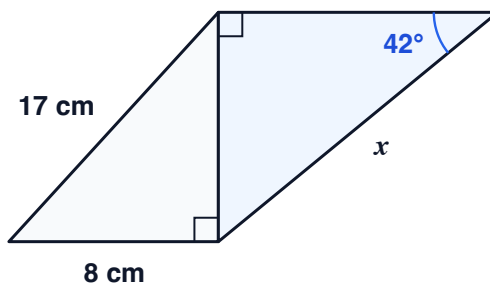
$$\tan(\theta) = \frac{\text{Opp}}{\text{Adj}} = \frac{18}{25} = 0.72$$

$$\theta = \tan^{-1}(0.72) \approx 35.8^\circ$$

Bearing is written as three digits: 036° .

EXAMPLE 4.2 **Calculator**

Two right-angled triangles share a common vertical side. Work out the value of x correct to 1 decimal place.

**STEP-BY-STEP SOLUTION:**

Step 1: Find the shared vertical side h using Pythagoras' theorem on the left triangle:

$$h = \sqrt{17^2 - 8^2} = \sqrt{289 - 64} = \sqrt{225} = 15 \text{ cm}$$

Step 2: In the right-hand triangle, 15 cm is Opposite the 42° angle and x is the Hypotenuse (SOH):

$$\sin(42^\circ) = \frac{15}{x} \implies x = \frac{15}{\sin(42^\circ)} \approx 22.4 \text{ cm}$$

Venn Diagrams

1 Set Notation

📖 CORE CONCEPT & METHOD

$A \cup B$ (**Union**): Everything in A OR B OR both.

$A \cap B$ (**Intersection**): Only the things in BOTH A AND B.

A' (**Complement**): Everything NOT in A.

Common Mistake: Always fill in the central intersection (BOTH / ALL THREE) first and SUBTRACT that overlap from the circle totals to find the "only" regions.

MODEL WORKED EXAMPLES

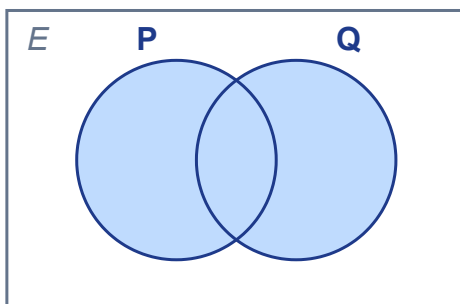
EXAMPLE 1.1

🚫 Non-Calculator

What does $P \cup Q$ mean?

STEP-BY-STEP SOLUTION:

$P \cup Q$ is the **Union** of sets P and Q. It means elements in P **OR** in Q (or in both).



Visual rule: Shade both entire circles.

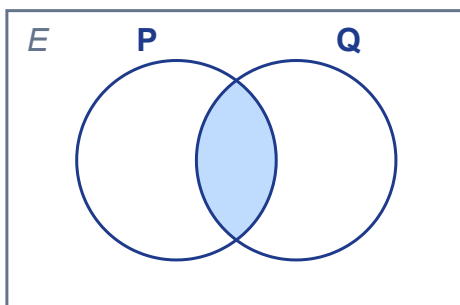
EXAMPLE 1.2

🚫 Non-Calculator

What does $P \cap Q$ mean?

STEP-BY-STEP SOLUTION:

$P \cap Q$ is the **Intersection** of sets P and Q. It means only the elements that are in P **AND** in Q.



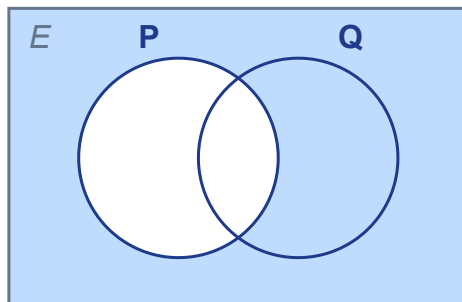
Visual rule: Shade only the overlapping region in the middle.

EXAMPLE 1.3 **Non-Calculator**

What does P' mean?

STEP-BY-STEP SOLUTION:

P' is the **Complement** of set P. It means everything that is **NOT** in P.



Visual rule: Shade everything inside the rectangle, but leave circle P completely blank.

2 2-Circle Venn Diagrams **CORE CONCEPT & METHOD**

When filling out a Venn Diagram from a word problem, ALWAYS start from the middle (the intersection) and work your way outwards!

If 20 people like Cats, and 5 like BOTH Cats and Dogs, then only 15 people like JUST Cats.

Common Mistake: Remember $A \cap B$ is the overlap ONLY (in both A and B), $A \cup B$ is everything inside either circle, and A' includes the outside region $\mathcal{E}$ as well.

MODEL WORKED EXAMPLES

EXAMPLE 2.1 **Non-Calculator**

In a group of 36 musicians, 22 play the Piano, 19 play the Guitar, and 4 play neither. How many play **both** instruments?

STEP-BY-STEP SOLUTION:

Step 1: Find the total number of musicians who play at least one instrument.

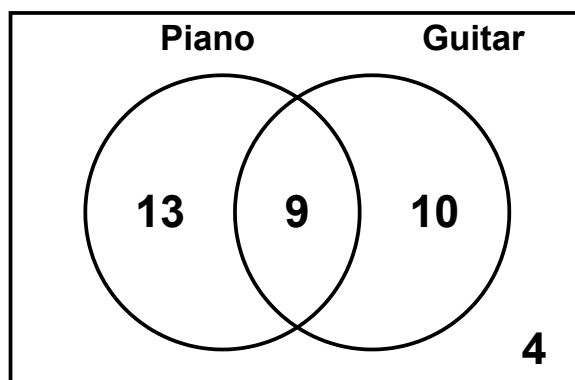
$$\text{Total} = 36 - 4 = 32$$

Step 2: Add the Piano and Guitar players together.

$$22 + 19 = 41$$

Step 3: The overlap (both) is the difference!

$$41 - 32 = 9 \text{ musicians play both.}$$

**EXAMPLE 2.2** **Non-Calculator**

There are 45 members in an outdoor club. 28 go Hiking, 21 go Kayaking, and 11 do BOTH. How many do **neither** activity?

STEP-BY-STEP SOLUTION:

Step 1: Always start with the intersection (both) in the middle of the Venn diagram: 11.

Step 2: Find the 'Hiking ONLY' and 'Kayaking ONLY' members.

$$\text{Hiking ONLY} = 28 - 11 = 17$$

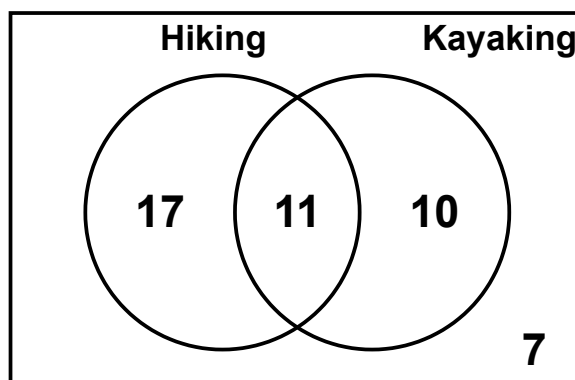
$$\text{Kayaking ONLY} = 21 - 11 = 10$$

Step 3: Add up everyone doing an activity.

$$17 + 11 + 10 = 38$$

Step 4: Find those who do neither.

$$45 - 38 = 7$$



3 3-Circle Venn Diagrams

□ CORE CONCEPT & METHOD

Start with the people who like ALL THREE things in the very centre.

Then, use subtraction to find the people who like exactly two things, and finally subtract again to find those who like exactly one thing.

Common Mistake: Work strictly from the centre outwards in a 3-circle Venn diagram: place "all three" first, subtract it from each pairwise overlap, then subtract all three inner pieces from each circle total.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

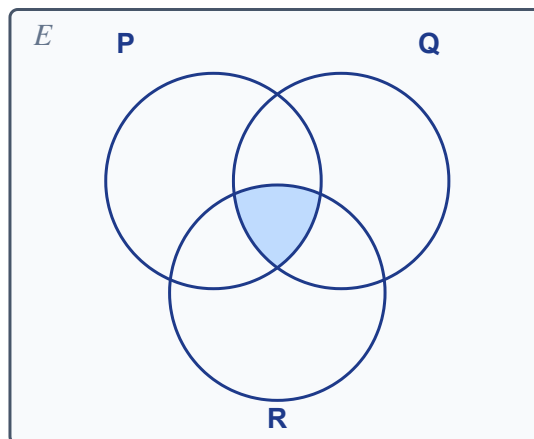
 **Non-Calculator**

A Venn diagram has 3 overlapping circles: P, Q, and R.

Where is $P \cap Q \cap R$ located?

STEP-BY-STEP SOLUTION:

It is the very centre region where **all three circles overlap**.



EXAMPLE 3.2 **Non-Calculator**

In a survey of 65 students about the clubs they attend:

- 4 students attend Drama, Robotics, AND Chess.
- 10 students attend Drama and Robotics.
- 9 students attend Drama and Chess.
- 11 students attend Robotics and Chess.
- 29 students attend Drama in total.
- 31 students attend Robotics in total.
- 27 students attend Chess in total.

Draw a Venn Diagram to find out how many students attend **NONE** of these clubs.

STEP-BY-STEP SOLUTION:

Step 1: Start in the very middle: 4 students attend all three.

Step 2: Subtract from the 'two clubs' totals.

$$\text{Drama \& Robotics ONLY} = 10 - 4 = 6$$

$$\text{Drama \& Chess ONLY} = 9 - 4 = 5$$

$$\text{Robotics \& Chess ONLY} = 11 - 4 = 7$$

Step 3: Subtract from the 'one club' totals.

$$\text{Drama ONLY} = 29 - (6 + 4 + 5) = 14$$

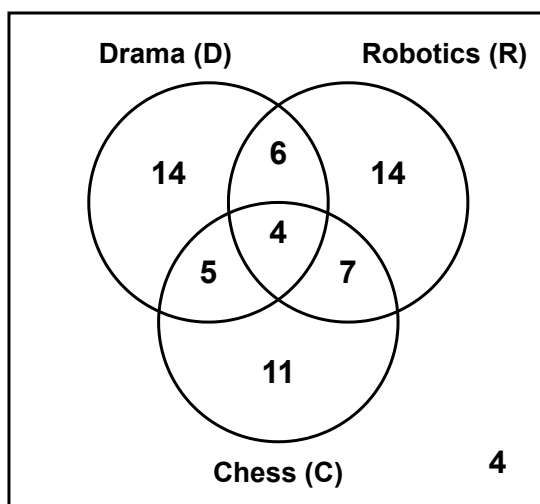
$$\text{Robotics ONLY} = 31 - (6 + 4 + 7) = 14$$

$$\text{Chess ONLY} = 27 - (5 + 4 + 7) = 11$$

Step 4: Find 'None' by subtracting everyone from 65.

$$\text{Total in circles} = 14 + 14 + 11 + 6 + 5 + 7 + 4 = 61$$

$$\text{None} = 65 - 61 = 4.$$



PROBABILITY — TREE DIAGRAMS

Probability Trees

1 Independent Events (AND)

□ CORE CONCEPT & METHOD

When two events happen one after the other, we use a probability tree diagram.

- Each pair of branches adds up to 1.
- Going along the branches ($\longrightarrow$): Multiply ($\times$)
- Going down the outcomes at the end ($\downarrow$): Add ($+$)

Common Mistake: Each pair of branches from a point must add to 1; MULTIPLY probabilities along a path (AND) and ADD separate path probabilities at the end (OR).

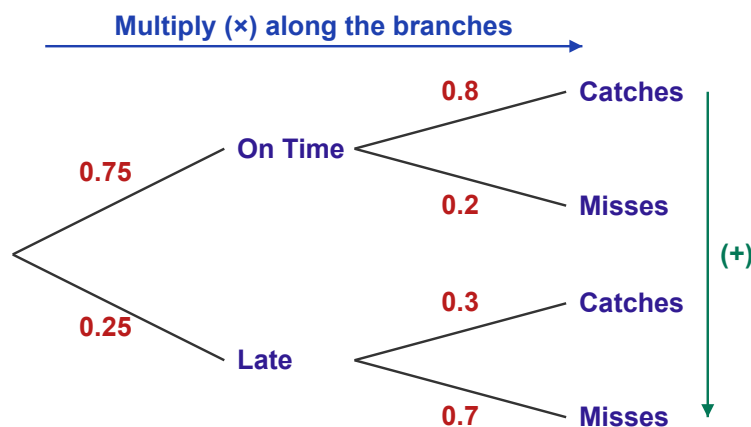
MODEL WORKED EXAMPLES

EXAMPLE 1.1

✎ Non-Calculator

The probability that a bus arrives on time is 0.75. If the bus is on time, the probability that Leo catches his train is 0.8. If the bus is late, the probability that Leo catches his train is 0.3.

Find the probability that the bus is on time AND Leo catches his train.



STEP-BY-STEP SOLUTION:

Multiply along the branches ($\longrightarrow$):

$$P(\text{On Time AND Catches}) = 0.75 \times 0.8 = \mathbf{0.6}$$

2 Combinations (OR)

▮ CORE CONCEPT & METHOD

If a question asks for 'exactly one win', it could happen two ways: (Win AND Lose) OR (Lose AND Win).

- **Multiply ($\times$)** along each branch path ($\longrightarrow$).
- **Add ($+$)** the separate branch outcomes together down the end ($\downarrow$).

Common Mistake: For "exactly one" or "same colour", list ALL matching two-step paths (e.g. Win–Lose AND Lose–Win), multiply along each path, and add those products together.

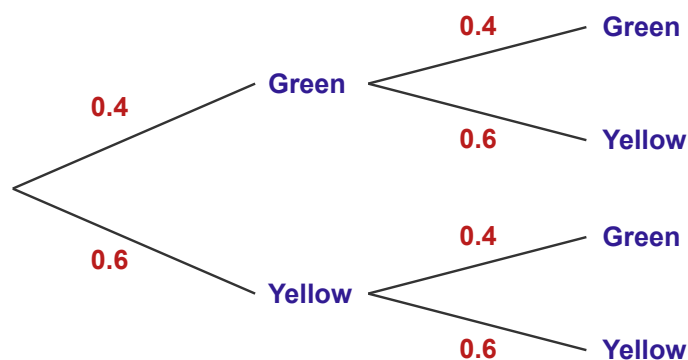
MODEL WORKED EXAMPLES

EXAMPLE 2.1

 **Non-Calculator**

A crate contains 4 green and 6 yellow apples. An apple is picked at random, replaced, and a second apple is picked.

Find the probability of picking two apples of the same colour.



STEP-BY-STEP SOLUTION:

Paths with the same colour: (Green, Green) or (Yellow, Yellow)

Multiply along the branches ($\longrightarrow$):

$$\text{Path 1 (Green, Green)} = 0.4 \times 0.4 = 0.16$$

$$\text{Path 2 (Yellow, Yellow)} = 0.6 \times 0.6 = 0.36$$

Add down the outcomes ($\downarrow$):

$$P(\text{Same colour}) = 0.16 + 0.36 = \mathbf{0.52}$$

3 At Least One

□ CORE CONCEPT & METHOD

If asked for the probability of being late on *at least one* day, you could add up 3 branches (Late/Late, Late/Not, Not/Late)...

OR you can use a shortcut! Find the probability of being late on ZERO days (Not/Not) and subtract it from 1.

$$P(\text{At least one}) = 1 - P(\text{None})$$

Common Mistake: For "at least one Win/Heads", use the complement shortcut $1 - P(\text{none})$ — multiply along the single "Lose then Lose" branch and subtract from 1.

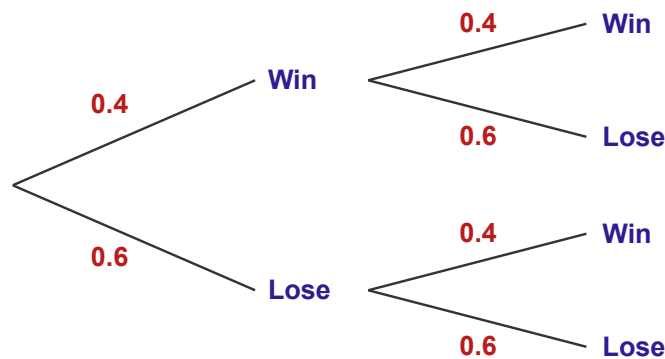
MODEL WORKED EXAMPLES

EXAMPLE 3.1

✎ Non-Calculator

A biased spinner has $P(\text{Win}) = 0.4$ and $P(\text{Lose}) = 0.6$. It is spun twice.

What is the probability of getting AT LEAST one Win?



STEP-BY-STEP SOLUTION:

Method (1 - P(None)):

The only path where you DON'T get at least one Win is losing both times (Lose, Lose).

$$P(\text{Lose, Lose}) = 0.6 \times 0.6 = 0.36$$

$$P(\text{At least one Win}) = 1 - 0.36 = \mathbf{0.64}$$