GCSE MATHEMATICS (9-1) — TUTORIALS & WORKED EXAMPLES

Grade 6 Mathematics — Tutorials & Examples

(Higher Tier)

Higher Tier Tutorials, Common Mistake Warnings & Step-by-Step Worked Examples

- **Higher Tier Core Concept Notes** & Precision Geometric/Graph Diagrams
- **Exam-Focused "Common Mistake" Warnings** on every single step
- **Model Worked Examples** tagged **Calculator** or **Non-Calculator**
- **Streamlined Higher Revision Guide** (Tutorials & Worked Examples Only)

Student Name: _____

Class / Teacher: _____

Target Grade: _____

SCAN FOR ONLINE PRACTICE & WORKED SOLUTIONS

BreakingDownMaths.com

Interactive Step-by-Step GCSE Mathematics Mastery

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NUMBER — FRACTIONS & DECIMALS

Recurring Decimals

1 Fractions to Recurring Decimals

CORE CONCEPT & METHOD

Some fractions don't divide perfectly—they go on forever! We call these **Recurring Decimals**.

Just use the bus stop method to divide the top by the bottom. When you notice the exact same remainder happening over and over, you know it's recurring! Write it with three dots at the end.

Common Mistake: Place recurring dots ONLY over the first and last digits of the repeating cycle (e.g. $0.\dot{1}42857\dot{}$), never over non-repeating digits or every middle digit.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

Write $\frac{5}{9}$ as a recurring decimal.

STEP-BY-STEP SOLUTION:

Use division: $5 \div 9$.

Put 5.000 inside the bus stop, and 9 outside:

Notice how the remainder **5** keeps repeating over and over? That means the decimal will go on forever!

So the answer is $0.555...$ which we write as $0.\dot{5}$

$$\frac{5}{9} = 0.\dot{5}$$

2 Recurring Decimals to Fractions

CORE CONCEPT & METHOD

To change a recurring decimal into a fraction, we use a clever algebra trick to make the repeating part **vanish**!

Call the decimal x. Multiply by 10 (or 100 if two numbers repeat) to shift the decimal point. Then subtract the original x from it!

> **Common Mistake:** When some decimal digits do NOT repeat (like $0.4\dot{7}$), multiply by two powers of 10 ($100x$ and $10x$) so the recurring tails match identically before subtracting — $0.4\dot{7} = \frac{43}{90}$, NEVER $\frac{47}{99}$.

MODEL WORKED EXAMPLES

EXAMPLE 2.1 Non-Calculator

Write $0.\dot{4}$ (which is $0.444...$) as a fraction.

STEP-BY-STEP SOLUTION:

1. Let $x = 0.444...$
2. Multiply by 10 to shift it: $10x = 4.444...$
3. Subtract them so the repeating parts perfectly align and cancel out:

$$\begin{aligned} 10x &= 4.444... \\ -\quad x &= 0.444... \\ \hline 9x &= 4 \end{aligned}$$

4. Divide by 9: $x = \frac{4}{9}$

$$0.\dot{4} = \frac{4}{9}$$

EXAMPLE 2.2 Non-Calculator

Write $0.\dot{3}\dot{6}$ (which is $0.3636\ldots$) as a fraction.

STEP-BY-STEP SOLUTION:

1. Let $x = 0.3636\ldots$
2. The repeating part is 2 digits long, so multiply by 100!
$100x = 36.3636\ldots$
3. Subtract them so the repeating parts perfectly align and cancel out:

$$\begin{array}{rl} 100x & = 36.3636\ldots \\ - \quad x & = \quad 0.3636\ldots \\ \hline 99x & = 36 \end{array}$$

4. Divide by 99: $x = \dfrac{36}{99}$
5. Simplify by dividing top and bottom by 9: $x = \dfrac{4}{11}$

$$0.\dot{3}\dot{6} = \frac{4}{11}$$

EXAMPLE 2.3 Non-Calculator

Write $0.3\dot{1}\dot{8}$ (which is $0.31818\ldots$) as a fraction.

STEP-BY-STEP SOLUTION:

1. Let $x = 0.31818\ldots$
2. The repeating part doesn't start immediately! Multiply by 10 to move the decimal to the start of the repeating part:
$10x = 3.1818\ldots$
3. Multiply by 1000 to move the decimal past the first repeating block:
$1000x = 318.1818\ldots$
4. Subtract them so the repeating parts perfectly align and cancel out:

$$\begin{array}{rl} 1000x & = 318.1818\ldots \\ - \quad 10x & = \quad 3.1818\ldots \\ \hline 990x & = 315 \end{array}$$

5. Divide by 990: $x = \dfrac{315}{990}$
6. Simplify: $x = \dfrac{7}{22}$

$$0.3\dot{1}\dot{8} = \frac{7}{22}$$

NUMBER — INDICES & STANDARD FORM

Negative & Fractional Indices

1 Negative & Fractional Indices

CORE CONCEPT & METHOD

Negative Indices mean '1 over' (the reciprocal). They do NOT make the number negative!

$$x^{-a} = \frac{1}{x^a}$$

Fractional Indices mean roots. The bottom of the fraction is the root, the top is the power.

$x^{1/2} = \sqrt{x}$ (Square root)
$x^{1/3} = \sqrt[3]{x}$ (Cube root)

Tip: If you have a fraction with a negative index, just flip the fraction upside down to make the power positive!

> **Common Mistake:** Take the ROOT first before raising to the power ($64^{-2/3} = (\sqrt[3]{64})^{-2} = 4^{-2} = \frac{1}{16}$), and remember a negative index means RECIPROCAL, never a negative answer.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

Evaluate 5^{-2}

STEP-BY-STEP SOLUTION:

1. The negative means '1 over': $\frac{1}{5^2}$
2. Apply the power: $5^2 = 25$

Answer: $\frac{1}{25}$

$$\frac{1}{25}$$

EXAMPLE 1.2

Non-Calculator

Evaluate $25^{3/2}$

STEP-BY-STEP SOLUTION:

1. The denominator 2 means square root: $\sqrt{25} = 5$
2. The numerator 3 means power of 3: $5^3 = 125$

Answer: 125

$$125$$

EXAMPLE 1.3 Non-Calculator

Evaluate $27^{-2/3}$

STEP-BY-STEP SOLUTION:

1. The negative means '1 over': $\frac{1}{27^{2/3}}$

2. Cube root of 27 is 3. Then $3^2 = 9$

Answer: $\frac{1}{9}$

$$\frac{1}{9}$$

EXAMPLE 1.4 Non-Calculator

Evaluate $\left(\frac{9}{16}\right)^{1/2}$

STEP-BY-STEP SOLUTION:

The power applies to BOTH the top and bottom.

Top: $\sqrt{9} = 3$

Bottom: $\sqrt{16} = 4$

Answer: $\frac{3}{4}$

$$\frac{3}{4}$$

EXAMPLE 1.5 Non-Calculator

Evaluate $\left(\frac{27}{64}\right)^{-2/3}$

STEP-BY-STEP SOLUTION:

1. The negative power flips the fraction: $\left(\frac{64}{27}\right)^{2/3}$

2. Cube root the top and bottom: $\frac{4}{3}$

3. Square the top and bottom: $\frac{16}{9}$

Answer: $\frac{16}{9}$

$$\frac{16}{9}$$

2 Solving Index Equations

CORE CONCEPT & METHOD

To find a missing power in an equation, use the index laws to simplify both sides into a **single base**.

If the bases are different (e.g. $1000^4 = 10^x$), you must rewrite the big number as a power of the small number first ($1000 = 10^3$)!

> **Common Mistake:** Rewrite every term with the SAME prime base before equating powers, and wrap subtracted exponents in brackets: $\frac{3^{2x}}{3^{x-1}} = 3^{2x-(x-1)} = 3^{x+1}$, NEVER 3^{x-1}.

MODEL WORKED EXAMPLES

EXAMPLE 2.1 — Non-Calculator

Given that $\frac{x^9}{x^a} = x^{13}$, find the value of a.

STEP-BY-STEP SOLUTION:

1. When dividing powers with the same base, subtract the indices: $9 - a = 13$
2. Rearrange to solve for a: $a = 9 - 13 = -4$

a = -4

EXAMPLE 2.2 — Non-Calculator

Given that $27^4 = 3^x$, find the value of x.

STEP-BY-STEP SOLUTION:

1. Rewrite 27 as a power of 3: $27 = 3^3$
2. Substitute into the equation: $(3^3)^4 = 3^x$
3. Multiply the powers: $3^{12} = 3^x$, so $x = 12$

x = 12

ALGEBRA — QUADRATICS

Expanding Triple Brackets

CORE CONCEPT & METHOD

For triple brackets, don't panic! Just expand the **last two brackets** first using FOIL or a 2x2 Grid.

Once simplified, use a **2x3 Grid Method** to multiply the quadratic by the first bracket!

Example: $(x+1)(x^2+5x+6)$

×	x^2	$+5x$	$+6$
x	x^3	$+5x^2$	$+6x$
$+1$	$+x^2$	$+5x$	$+6$

Then just gather your like terms diagonally!

$+x^2 + 5x^2 = +6x^2$

$+5x + 6x = +11x$

So the final expanded expression is:

$$x^3 + 6x^2 + 11x + 6$$

Common Mistake: Expand ONLY two brackets first and simplify the quadratic completely before multiplying every term by the third bracket — never multiply just the first terms and constants.

MODEL WORKED EXAMPLES

EXAMPLE 1

Non-Calculator

Expand and simplify $(x+2)(x+3)(x+4)$

STEP-BY-STEP SOLUTION:

1. Ignore $(x+2)$ for now. Expand $(x+3)(x+4)$:
$x^2 + 4x + 3x + 12 = x^2 + 7x + 12$

2. Now multiply by $(x+2)$:
$(x+2)(x^2+7x+12)$

Multiply everything by x: $x^3 + 7x^2 + 12x$
Multiply everything by 2: $+2x^2 + 14x + 24$

3. Collect like terms:

$$x^3 + 9x^2 + 26x + 24$$

EXAMPLE 2

Non-Calculator

Expand and simplify $(2x+3)(3x-1)(4-x)$

STEP-BY-STEP SOLUTION:

1. Expand the first two brackets $(2x+3)(3x-1)$:
$6x^2 - 2x + 9x - 3 = 6x^2 + 7x - 3$

2. Now multiply $(6x^2+7x-3)$ by $(4-x)$:
Multiply everything by 4: $24x^2 + 28x - 12$
Multiply everything by $-x$: $-6x^3 - 7x^2 + 3x$

3. Collect like terms in descending powers of x:

$$-6x^3 + 17x^2 + 31x - 12$$

ALGEBRA — EQUATIONS & SEQUENCES

Graphical Inequalities

CORE CONCEPT & METHOD

Inequalities can be represented on a graph. The line acts as a boundary fence!

- $y \geq \ldots$ means the region is **ABOVE** the line.
- $y \leq \ldots$ means the region is **BELOW** the line.
- $x \geq \ldots$ means the region is **TO THE RIGHT** of the vertical line.

If an inequality has a strict $<$ or $>$, we draw a **dashed line**. If it has $\leq$ or $\geq$, we draw a **solid line**.

> **Common Mistake:** Draw a DASHED line for strict inequalities ($<$ or $>$) and a SOLID line for inclusive inequalities ($\leq$ or $\geq$); integer points lying on a dashed boundary do NOT satisfy the inequality.

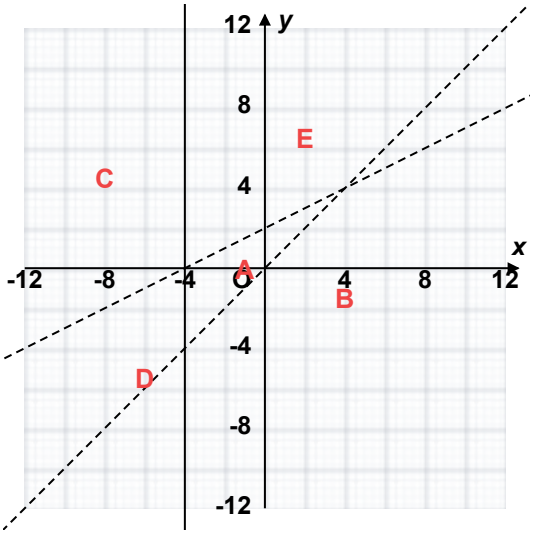
MODEL WORKED EXAMPLES

EXAMPLE 1

Non-Calculator

Which region (A, B, C, D, or E) satisfies the inequalities:

$x \geq -4$
$y > x$
$y < 0.5x + 2$



STEP-BY-STEP SOLUTION:

1. $x \geq -4$ means we must be to the right of the **solid** vertical line $x = -4$.
2. $y > x$ means we must be above the **dashed** diagonal line $y = x$.
3. $y < 0.5x + 2$ means we must be below the shallower **dashed** line $y = 0.5x + 2$.

The only region trapped between all three is **A**.

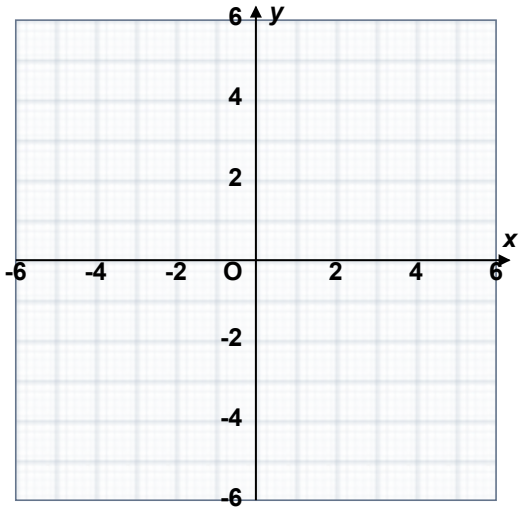
Region A

EXAMPLE 2

Non-Calculator

On the grid below, draw the boundary lines and shade the region R that satisfies all three inequalities:

$$x \geq -1, \quad y > 1, \quad x + y < 4$$

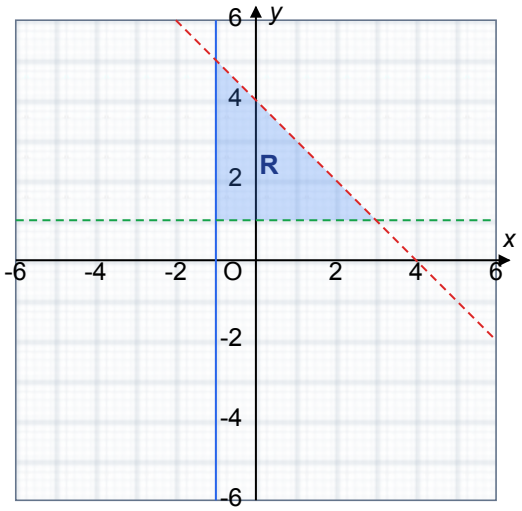


STEP-BY-STEP SOLUTION:

1. Plot the three boundary lines (use a **solid** line for $\geq$ and **dashed** lines for strict $>$ and $<$):

- $x = -1$: **solid** vertical line through $x = -1$ ($x \geq -1$ is to the **right**).
- $y = 1$: **dashed** horizontal line through $y = 1$ ($y > 1$ is **above**).
- $x + y = 4$: **dashed** line passing through $(0, 4)$ and $(4, 0)$ ($x + y < 4$ is **below**).

2. Shade the region R satisfying all three inequalities:



Region R shaded

ALGEBRA — COORDINATE GEOMETRY

Perpendicular Lines

1 Parallel Lines

CORE CONCEPT & METHOD

Parallel lines have the **exact same gradient**. If line A is $y = 3x - 2$, any parallel line will start with $y = 3x + c$.

> **Common Mistake:** Rearrange any equation into $y = mx + c$ FIRST before reading off the gradient m — in $x + 3y = 15$, the gradient is $-\frac{1}{3}$, NOT 1 or 3.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

Find the equation of a line parallel to $y = 7x - 2$ passing through $(0, 8)$.

STEP-BY-STEP SOLUTION:

Gradient $m = 7$.
y-intercept $c = 8$.
$y = 7x + 8$.

$$y = 7x + 8$$

EXAMPLE 1.2

Non-Calculator

Find the equation of a line parallel to $y = 5x - 3$ passing through $(2, 16)$.

STEP-BY-STEP SOLUTION:

Parallel lines have the same gradient, so $m = 5$.
This gives us part of the equation: $y = 5x + c$.

Substitute the x and y values from the point $(2, 16)$ to find c:
$16 = 5(2) + c$
$16 = 10 + c$
$c = 6$

Equation: $y = 5x + 6$.

$$y = 5x + 6$$

2 Perpendicular Gradients

CORE CONCEPT & METHOD

Perpendicular lines cross at $90°$. Their gradients are **negative reciprocals** of each other. (Flip the fraction and flip the sign!). If $m_1 = 3$, $m_2 = -\frac{1}{3}$.

> **Common Mistake:** Perpendicular gradients multiply to -1 ($m_\perp = -\frac{1}{m}$): flip the fraction AND flip the sign (so gradient $\frac{3}{2}$ becomes $-\frac{2}{3}$, never $+\frac{2}{3}$ or $-\frac{3}{2}$).

MODEL WORKED EXAMPLES

EXAMPLE 2.1 Non-Calculator

Line A has gradient 5. Line B is perpendicular to Line A. Find its gradient.

STEP-BY-STEP SOLUTION:

Flip 5 (which is $5/1$) to get $1/5$. Flip the sign to make it negative.

$m = -\frac{1}{5}$.

$$-\frac{1}{5}$$

EXAMPLE 2.2 Non-Calculator

Find the equation of the line perpendicular to $y = \frac{1}{6}x - 1$ which passes through $(0, 7)$.

STEP-BY-STEP SOLUTION:

1. The gradient of the original line is $\frac{1}{6}$.

2. To find the perpendicular gradient, flip the fraction to $\frac{6}{1}$ (which is 6) and flip the sign.

$m = -6$

3. It passes through $(0, 7)$, meaning the y-intercept $c = 7$.

Equation: $y = -6x + 7$.

$$y = -6x + 7$$

EXAMPLE 2.3 Non-Calculator

Find the equation of the line perpendicular to $2y = 5x + 8$ which passes through $(0, -3)$.

STEP-BY-STEP SOLUTION:

1. **Rearrange** the first equation to find its gradient:

$2y = 5x + 8$

$y = \frac{5}{2}x + 4$

Gradient $= \frac{5}{2}$

2. Find the **perpendicular gradient** (flip fraction and sign):

$m = -\frac{2}{5}$

3. It passes through $(0, -3)$, so $c = -3$.

Equation: $y = -\frac{2}{5}x - 3$.

$$y = -\frac{2}{5}x - 3$$

3 Perpendicular Bisectors

CORE CONCEPT & METHOD

A **perpendicular bisector** of AB cuts through the **midpoint** of AB at $90°$.

1. Find the **midpoint** $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
2. Find the gradient of AB, then take the **negative reciprocal** for the perpendicular gradient $m_\perp$.
3. Substitute M into $y = m_\perp x + c$ to find c.

> **Common Mistake:** Substitute the MIDPOINT of AB (never point A or B) together with the PERPENDICULAR gradient $-\frac{1}{m_{AB}}$ when finding c for a perpendicular bisector.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

Non-Calculator

Find the midpoint of $(4, 6)$ and $(10, 14)$.

STEP-BY-STEP SOLUTION:

x: $(4 + 10)/2 = 7$
y: $(6 + 14)/2 = 10$
Midpoint: $(7, 10)$.

$$(7, 10)$$

EXAMPLE 3.2

Non-Calculator

Find the equation of the perpendicular bisector of the line segment joining $A(2, 3)$ and $B(6, 11)$.

STEP-BY-STEP SOLUTION:

1. **Midpoint** of AB: $M = \left(\frac{2+6}{2}, \frac{3+11}{2}\right) = (4, 7)$.
2. **Gradient** of AB: $m_{AB} = \frac{11-3}{6-2} = \frac{8}{4} = 2$.
3. **Perpendicular gradient**: $m_\perp = -\frac{1}{2}$.
4. Substitute $(4, 7)$ into $y = -\frac{1}{2}x + c$:

$7 = -\frac{1}{2}(4) + c \implies 7 = -2 + c \implies c = 9$.

Equation: $y = -\frac{1}{2}x + 9$.

$$y = -\frac{1}{2}x + 9$$

GEOMETRY AND MEASURES — CIRCLES

Circle Theorems

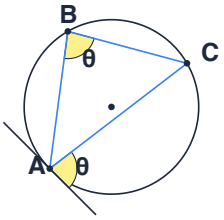
1 Theorems Cheat Sheet & Practice

CORE CONCEPT & METHOD

Complete Circle Theorems Reference

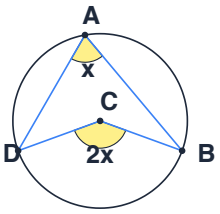
Review all 9 theorems before starting the mixed practice below.

Alternate segment theorem



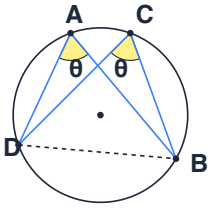
The angle that lies between a tangent and a chord is equal to the angle subtended by the same chord in the alternate segment.

Angle at the centre theorem



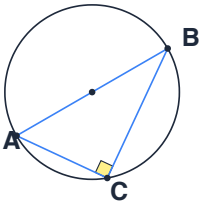
The angle at the centre is twice the angle at the circumference.

Angles in the same segment



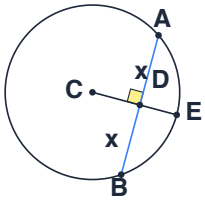
Angles in the same segment are equal.

Angles in a semicircle



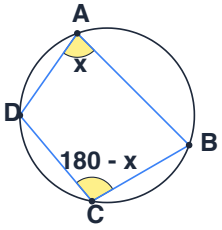
The angle in a semicircle is 90 degrees.

Chord of a circle



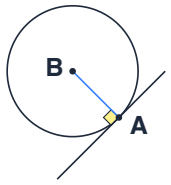
The perpendicular from the centre to a chord bisects the chord.

Cyclic quadrilateral



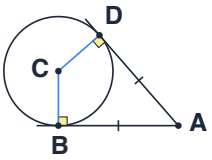
Opposite angles in a cyclic quadrilateral total 180°.

Tangent to a circle



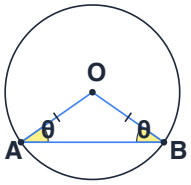
The angle between a tangent and radius is 90 degrees.

Two tangents



Tangents which meet at the same point are equal in length.

Isosceles triangle (Radii)



A triangle formed by two radii and a chord is an isosceles triangle. Therefore, the base angles are equal.

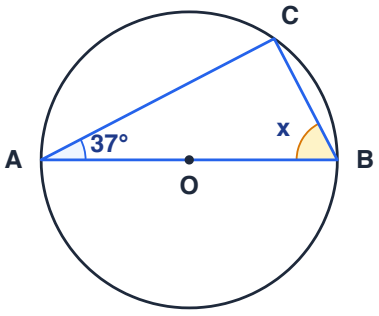
Common Mistake: When asked for geometric reasons, use exact Edexcel wording (e.g. "Alternate segment theorem" or "Opposite angles of a cyclic quadrilateral sum to $180°$"), never vague phrases like "circle rule".

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

Points A, B and C lie on the circumference of a circle with centre O. AB is a diameter and angle $CAB = 37°$.



Work out the size of angle ABC (x).

STEP-BY-STEP SOLUTION:

1. The angle in a semicircle is $90°$, so $\angle ACB = 90°$.
2. Angles in $\triangle ABC$ add up to $180°$:

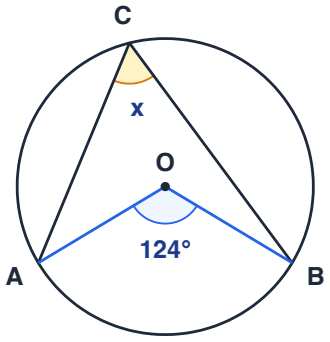
$$x = 180° - 90° - 37° = 53°$$

$$53°$$

EXAMPLE 1.2

Non-Calculator

A, B and C are points on the circumference of a circle with centre O.
Angle $AOB = 124°$.



Find the size of angle ACB (x).

STEP-BY-STEP SOLUTION:

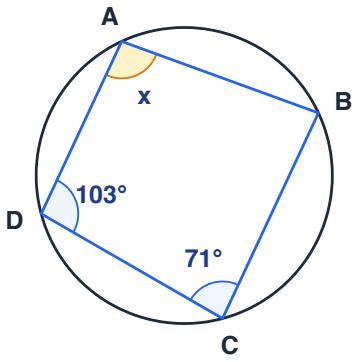
1. The angle at the centre is twice the angle at the circumference.
2. $\angle ACB = 124° \div 2 = 62°$.

$$62°$$

EXAMPLE 1.3

Non-Calculator

$ABCD$ is a cyclic quadrilateral inscribed in a circle.
Angle $ADC = 103°$ and angle $BCD = 71°$.



Calculate the size of angle DAB (x).

STEP-BY-STEP SOLUTION:

1. Opposite angles in a cyclic quadrilateral add up to $180°$.
2. Angle DAB is opposite angle BCD ($71°$):

$$x = 180° - 71° = 109°$$

$$109°$$

2 Mixed Theorem Practice

CORE CONCEPT & METHOD

Solving Multi-Step Circle Theorem Problems

In GCSE exam questions, you often need to combine two or more circle theorems with basic angle facts:

- **Isosceles Triangles from Radii:** Always check if two sides of a triangle are radii ($OA = OB$). If so, the base angles are equal!
- **Tangents & Radii (90°):** When a tangent meets a radius, mark the 90° angle immediately. Two tangents from the same point have equal length and form a kite with two 90° angles.
- **Parallel Chords or Tangents:** Look for alternate (Z), corresponding (F), or co-interior (C) angles alongside the Alternate Segment Theorem.

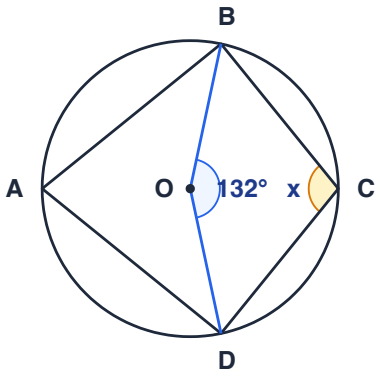
Common Mistake: Spot equal radii ($OA = OB$) forming hidden isosceles triangles inside the circle, and check all FOUR vertices lie on the circumference before using the 180° cyclic quadrilateral rule.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

Non-Calculator

A, B, C and D are points on the circumference of a circle with centre O.
Angle $BOD = 132^\circ$.



Work out the size of angle BCD (x).

STEP-BY-STEP SOLUTION:

1. The angle at the centre is twice the angle at the circumference:

$$\angle BAD = 132^\circ \div 2 = 66^\circ$$

2. Opposite angles in cyclic quadrilateral $ABCD$ add up to 180°:

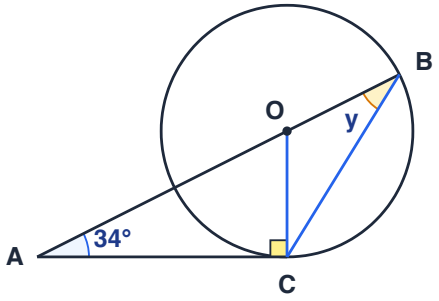
$$\angle BCD = 180^\circ - 66^\circ = 114^\circ$$

$$114^\circ$$

EXAMPLE 2.2

Non-Calculator

AOB is a straight line passing through the centre O.
B and C lie on the circumference, and AC is a tangent to the circle at C.
Angle $CAB = 34°$.



Work out the size of angle y.

STEP-BY-STEP SOLUTION:

1. The angle between tangent AC and radius OC is $90°$, so $\angle OCA = 90°$.
2. In $\triangle AOC$, $\angle AOC = 180° - 90° - 34° = 56°$.
3. Since AOB is a straight line, $\angle BOC = 180° - 56° = 124°$.
4. Since $OB = OC$ (radii), $\triangle OBC$ is isosceles:

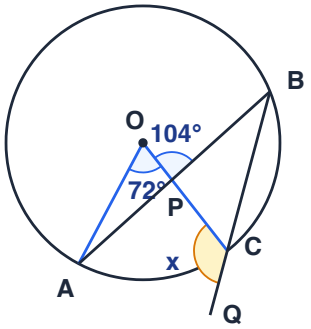
$$y = \frac{180° - 124°}{2} = 28°$$

$$28°$$

EXAMPLE 2.3

Non-Calculator

A, B and C are points on the circumference of a circle, centre O.
The lines OC and AB meet at the point P, and BCQ is a straight line.
Angle $AOC = 72°$ and angle $OPB = 104°$.



Work out the size of angle OCQ (x).

STEP-BY-STEP SOLUTION:

1. The angle at the centre is twice the angle at the circumference:

$$\angle ABC = 72° \div 2 = 36°$$

2. In $\triangle PBC$, $\angle PCB = 180° - 104° - 36° = 40°$.
3. Since BCQ is a straight line:

$$\angle OCQ = 180° - 40° = 140°$$

$$140°$$

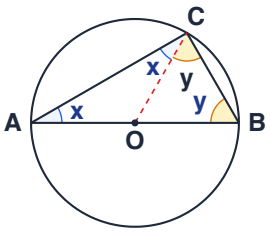
3 Circle Theorem Proofs

CORE CONCEPT & METHOD

Formal Circle Theorem Proofs

In GCSE Higher exams, you must be able to prove the circle theorems by drawing auxiliary radii (dashed red lines) to create isosceles triangles.

1. Prove the angle in a semi-circle is 90°



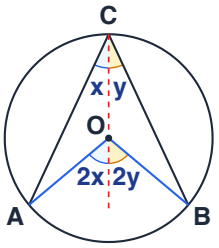
1. Draw radius OC from the centre O. Since $OA = OB = OC$ (radii), $\triangle OAC$ and $\triangle OBC$ are isosceles.
2. Let $\angle OAC = x$ (so $\angle OCA = x$) and $\angle OBC = y$ (so $\angle OCB = y$).
3. Angles in $\triangle ABC$ sum to $180°$:

$$x + (x + y) + y = 180°$$

$$2x + 2y = 180° \implies x + y = 90°$$

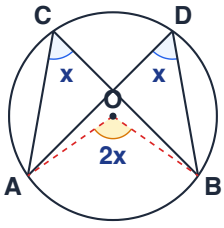
Therefore, $\angle ACB = 90°$.

2. Prove angle at centre is twice angle at circumference



1. Draw a line from C through the centre O, splitting $\angle ACB$ into x and y.
2. Since $OA = OC = OB$ (radii), $\triangle OAC$ and $\triangle OBC$ are isosceles, so $\angle OAC = x$ and $\angle OBC = y$.
3. The exterior angles at O are $2x$ and $2y$.
4. Total angle at the centre $= 2x + 2y = 2(x + y) = 2 \times \angle ACB$.

3. Prove angles in the same segment are equal

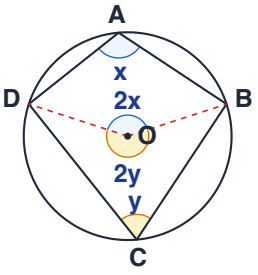


1. Join points A and B to the centre O, and let $\angle ACB = x$.
2. Since the angle at the centre is twice the angle at the circumference, $\angle AOB = 2x$.
3. Applying the same theorem to $\angle ADB$:

$$\angle ADB = \frac{1}{2}\angle AOB = \frac{1}{2}(2x) = x$$

Therefore, $\angle ACB = \angle ADB$.

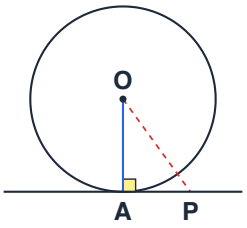
4. Prove opposite angles in a cyclic quad add to 180°



1. Join vertices B and D to the centre O. Let $\angle BAD = x$ and $\angle BCD = y$.
2. Since the angle at the centre is twice the angle at the circumference, obtuse $\angle BOD = 2x$ and reflex $\angle BOD = 2y$.
3. Angles around point O sum to $360°$:

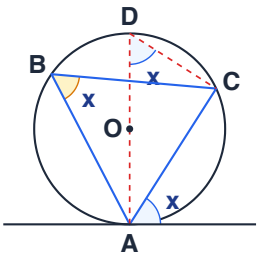
$$2x + 2y = 360° \implies x + y = 180°$$

5. Prove angle between tangent and radius is 90°



1. The tangent touches the circle only at A; any other point P on the tangent lies outside the circle.
2. Therefore, $OP > OA$ for every point $P \neq A$, meaning OA is the shortest distance from O to the line.
3. Since the shortest distance from a point to a line is perpendicular to the line, $\angle OAP = 90^\circ$.

6. Prove the Alternate Segment Theorem



1. Let the angle between the tangent and chord AC be x. Draw diameter AD through O and join CD.
2. Since tangent $\perp$ radius, $\angle DAC = 90^\circ - x$.
3. Since AD is a diameter, $\angle ACD = 90^\circ$, so $\angle ADC = 180^\circ - 90^\circ - (90^\circ - x) = x$.
4. By angles in the same segment, $\angle ABC = \angle ADC = x$.

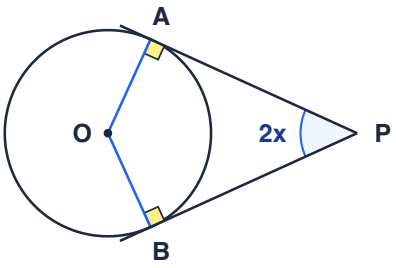
Common Mistake: In algebraic circle proofs, never assume the angle you are asked to prove — deduce each angle step by step from the given x and write a geometric reason beside every line.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

Non-Calculator

P is an external point. PA and PB are tangents to a circle with centre O, touching the circle at A and B. Angle $APB = 2x$.
Prove algebraically that angle $AOB = 180^\circ - 2x$, giving a geometric reason for each stage of your working.



STEP-BY-STEP SOLUTION:

Step 1: $\angle OAP = 90^\circ$ and $\angle OBP = 90^\circ$ *(a tangent meets a radius at 90°)*.

Step 2: In quadrilateral $OAPB$, the interior angles sum to 360° *(angles in a quadrilateral sum to 360°)*:

$$\angle AOB + 90^\circ + 2x + 90^\circ = 360^\circ$$

Step 3: Simplify and rearrange:

$$\angle AOB + 180^\circ + 2x = 360^\circ \implies \angle AOB = 180^\circ - 2x \quad \text{(proven)}$$

GEOMETRY AND MEASURES — TRANSFORMATIONS & VECTORS

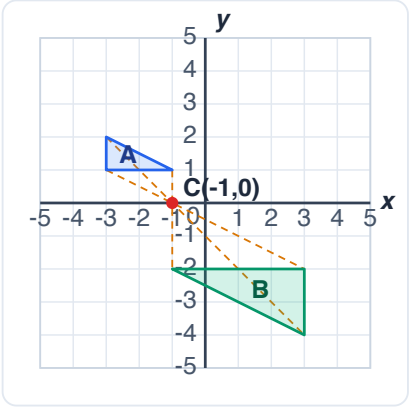
Negative Enlargements & Invariant Points

CORE CONCEPT & METHOD

Negative Scale Factors, Combined Transformations & Invariant Points

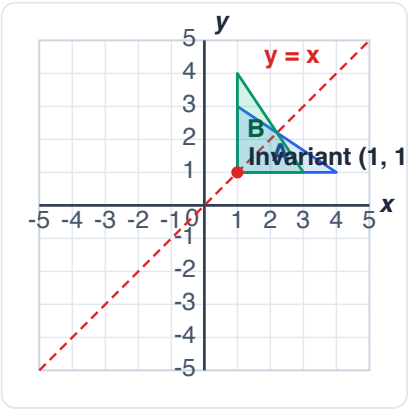
In Higher GCSE Maths, enlargements can have **negative scale factors**, and transformations can be chained together to produce **invariant points** (points that remain in the exact same position).

1. Negative Scale Factors ($k < 0$)



• Multiply the vector $\vec{CP}$ from the centre C to each vertex P by the negative scale factor k: $\vec{CP'} = k\vec{CP}$.
• Rays pass **through C to the opposite side**, turning the image upside-down!
• Note: $k = -1$ is identical to a $180°$ rotation about C.

2. Combined Transformations & Invariant Points



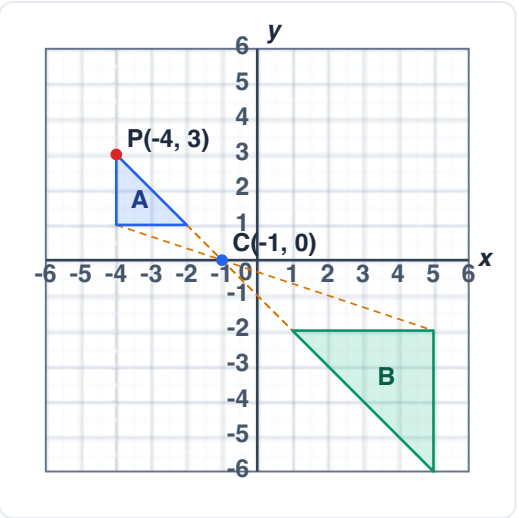
• **Combined transformations:** Apply the first transformation to find the intermediate shape, then apply the second.
• **Invariant point:** A point whose coordinates do not change ($P' = P$), such as a vertex lying directly on a mirror line or at a centre of rotation/enlargement.

Common Mistake: Count every column vector FROM the centre of enlargement C (never from the origin unless $C = (0, 0)$), and remember a negative scale factor sends each vertex through C to the opposite side.

MODEL WORKED EXAMPLES

EXAMPLE 1 Non-Calculator

Triangle A is enlarged by scale factor -2 about the centre $C(-1, 0)$ to give Triangle B, as shown on the coordinate grid below.



Verify the coordinates of the image of vertex $P(-4, 3)$.

STEP-BY-STEP SOLUTION:

Step 1: Find the vector from the centre $C(-1, 0)$ to $P(-4, 3)$

$$\vec{CP} = \begin{pmatrix} -4-(-1) \\ 3-0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

Step 2: Multiply by the negative scale factor -2 and add to $C(-1, 0)$

$$\vec{CP'} = -2\begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

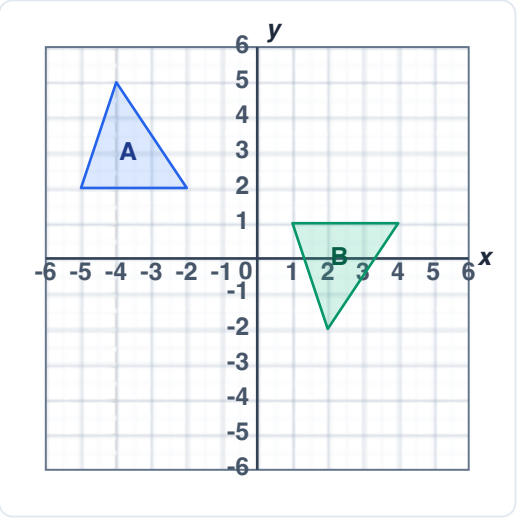
$$P' = (-1+6,\ 0+(-6)) = (5, -6)$$

$$(5, -6)$$

EXAMPLE 2

Non-Calculator

Triangle A and Triangle B are drawn on the coordinate grid below.



Triangle A is reflected in the x-axis and then translated by the column vector $\begin{pmatrix} p \\ q \end{pmatrix}$ to give Triangle B.

Find the values of p and q, giving your answer as (p, q).

STEP-BY-STEP SOLUTION:

Step 1: Reflect Triangle A in the x-axis ($y = 0$)

Pick the top vertex $(-4, 5)$ on A. Reflecting in the x-axis maps $(-4, 5) \mapsto (-4, -5)$.

Step 2: Find the translation from $(-4, -5)$ to the matching apex $(2, -2)$ on B

$$p = 2 - (-4) = 6$$
$$q = -2 - (-5) = 3$$

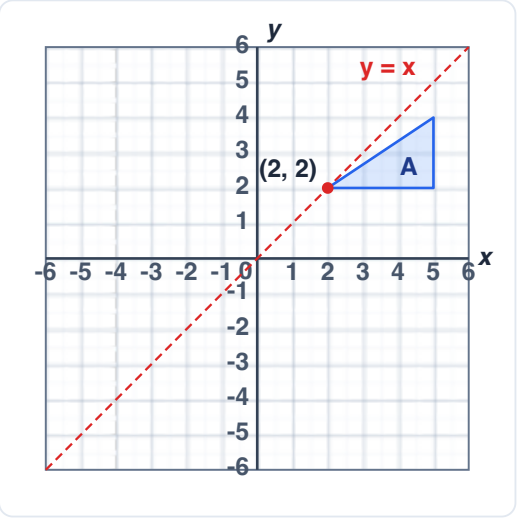
So the translation vector is $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$, or $(6, 3)$.

$$(6, 3)$$

EXAMPLE 3

Non-Calculator

Triangle A has vertices $(2, 2)$, $(5, 2)$, and $(5, 4)$, as shown on the coordinate grid below.



Triangle A is reflected in the line $y = x$.
Write down the coordinates of the invariant vertex on Triangle A.

STEP-BY-STEP SOLUTION:

Step 1: Recall when a point is invariant under a reflection

A point is invariant under a reflection if and only if it lies directly on the mirror line.

Step 2: Check which vertex satisfies $y = x$

Of the three vertices $(2, 2)$, $(5, 2)$, and $(5, 4)$, only $(2, 2)$ has $y = x$, so $(2, 2) \mapsto (2, 2)$.

$$(2, 2)$$

PROBABILITY — COMBINATIONS

Product Rule for Counting

1 Working Backwards

CORE CONCEPT & METHOD

If you know the total number of combinations, you can work backwards by **dividing**!

If there are 5 starters and x mains, and 40 combinations in total, then $5 \times x = 40$.
So $x = 40 \div 5 = 8$ mains.

> **Common Mistake:** Check whether options are combined by AND (multiply: $a \times b$) or OR (add separate cases: $ab + bc$) before setting up your equation for the unknown.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

A bakery offers 6 cake flavours and x frosting types. There are 54 possible combinations of one cake flavour and one frosting type. Find the value of x.

STEP-BY-STEP SOLUTION:

Using the product rule:

$$6 \times x = 54$$

Divide both sides by 6:

$$x = 54 \div 6 = 9$$

So there are **9 frosting types** available.

2 Permutations (Order Matters)

CORE CONCEPT & METHOD

If you have 50 cards and give one to Alice and one to Bob, the order matters! (Alice getting the Ace is different to Bob getting the Ace).

You have 50 choices for Alice, and 49 left for Bob. Total ways = 50×49.

> **Common Mistake:** When order matters and repeats are forbidden, deal with restricted slots (such as an odd first digit) FIRST, then reduce the remaining pool by 1 for each subsequent slot.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

Calculator

There are 10 finalists in a photography contest. In how many different ways can the Gold, Silver and Bronze prizes be awarded?

STEP-BY-STEP SOLUTION:

The order matters — winning Gold is different to winning Silver!

- There are **10** choices for the Gold prize.
- Once Gold is decided, there are **9** finalists left for the Silver prize.
- Then there are **8** finalists left for the Bronze prize.

Total arrangements:

$$10 \times 9 \times 8 = 720$$

3 Combinations (Order Doesn't Matter)

CORE CONCEPT & METHOD

If 10 people shake hands with each other, Person A shaking with Person B is the EXACT SAME as Person B shaking with Person A.

You must divide by 2 to remove the duplicates!

$$\text{Handshakes} = \frac{10 \times 9}{2}$$

Common Mistake: When order does NOT matter (selecting a pair, handshakes, or single round-robin matches), divide $n(n-1)$ by 2 so each pair (A, B) and (B, A) is counted once.

MODEL WORKED EXAMPLES

EXAMPLE 3.1 Calculator

There are 9 chess players in a tournament. Every player plays one match against each of the other players. How many matches are played in total?

STEP-BY-STEP SOLUTION:

Order does not matter — Player A playing Player B is the same match as Player B playing Player A.

Divide by 2 to remove duplicates:

$$\frac{9 \times 8}{2} = \frac{72}{2} = 36$$

So **36 matches** are played in total.

STATISTICS — AVERAGES & SPREAD

Cumulative Frequency

1 Cumulative Tables & Plotting

CORE CONCEPT & METHOD

Cumulative Frequency means a running total. You just add up the frequencies as you go down the table.

When plotting the graph, you ALWAYS plot the cumulative frequency against the **UPPER BOUND** of the class interval!

> **Common Mistake:** ALWAYS plot cumulative frequency against the UPPER BOUND of each class interval (never the midpoint!), and join plotted crosses with a smooth increasing S-curve.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

The table shows data about the lengths (mm) of seed pods.

Length (mm)	Frequency	Cumulative Frequency
$0 < l \leq 30$	10	10
$30 < l \leq 60$	24	?
$60 < l \leq 90$	16	?

Complete the cumulative frequency column. What point would be plotted for the interval $30 < l \leq 60$?

STEP-BY-STEP SOLUTION:

Step 1 — Running totals:

Row 1: 10

Row 2: $10 + 24 = 34$

Row 3: $34 + 16 = 50$

Step 2 — Identify the plot point for $30 < l \leq 60$:

Always plot against the **upper bound**. Upper bound = 60, cumulative frequency = 34.

Answer: Plot the point at $(60,\ 34)$.

EXAMPLE 1.2

Non-Calculator

Here is the completed table from the previous example.

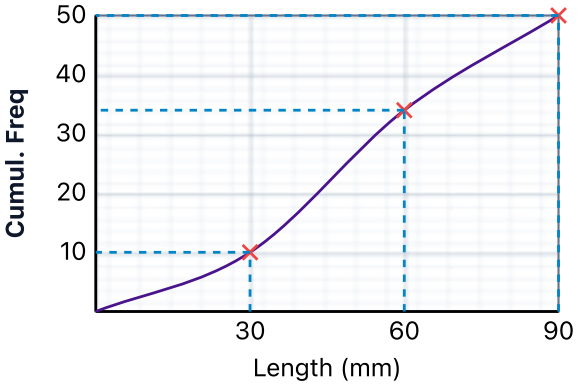
Length (mm)	Frequency	Cumulative Frequency
0 < l ≤ 30	10	10
30 < l ≤ 60	24	34
60 < l ≤ 90	16	50

Plot the cumulative frequency graph.

STEP-BY-STEP SOLUTION:

Step 1 – ALWAYS plot the cumulative frequency against the **Upper Bound** of each interval!
The coordinates to plot are:
• Interval $0 < l \leq 30$ → plot **(30, 10)**
• Interval $30 < l \leq 60$ → plot **(60, 34)**
• Interval $60 < l \leq 90$ → plot **(90, 50)**

Step 2 – Plot these points with crosses and join them with a smooth curve (starting from 0 if possible).



2 Estimating Median & IQR

CORE CONCEPT & METHOD

To find the **Median**, go to half the total frequency on the y-axis, and read across to the x-axis.

- Lower Quartile (LQ) is at $1/4$ of the total.
- Upper Quartile (UQ) is at $3/4$ of the total.
- **Interquartile Range (IQR)** = UQ - LQ.

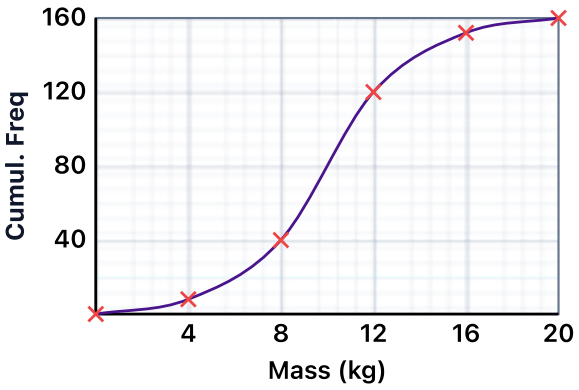
> **Common Mistake:** Locate quartiles on the vertical cumulative frequency axis at $\frac{N}{4}$, $\frac{N}{2}$, and $\frac{3N}{4}$ using the total data count N (not half the top axis label if the grid extends higher).

MODEL WORKED EXAMPLES

EXAMPLE 2.1

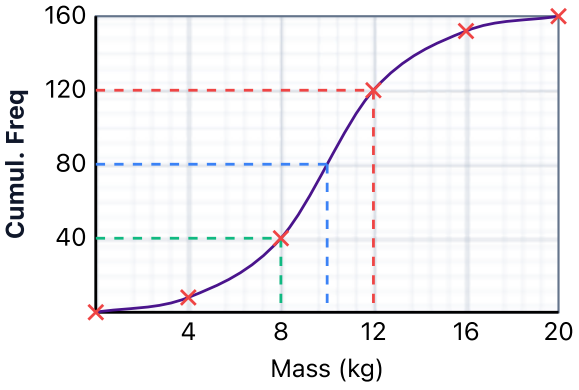
Non-Calculator

The cumulative frequency graph shows the masses (kg) of 160 parcels.



Estimate the Median, Lower Quartile (LQ), Upper Quartile (UQ), and the Interquartile Range (IQR).

STEP-BY-STEP SOLUTION:



Step 1 – Median: $160 \div 2 = 80$. Draw a line from $y = 80$ to the curve, and down. Median = 10 kg.

Step 2 – Lower Quartile (LQ): $160 \div 4 = 40$. Draw a line from $y = 40$. LQ = 8 kg.

Step 3 – Upper Quartile (UQ): $160 \times 0.75 = 120$. Draw a line from $y = 120$. UQ = 12 kg.

Step 4 – IQR: $UQ - LQ = 12 - 8 = 4$ kg.

3 Thresholds (More / Less)

CORE CONCEPT & METHOD

If a question asks how many people took **less than** 80 seconds, you just read the y-value at x=80.

If it asks how many took **MORE than** 80 seconds, you read the y-value at x=80, and **subtract it from the TOTAL frequency**!

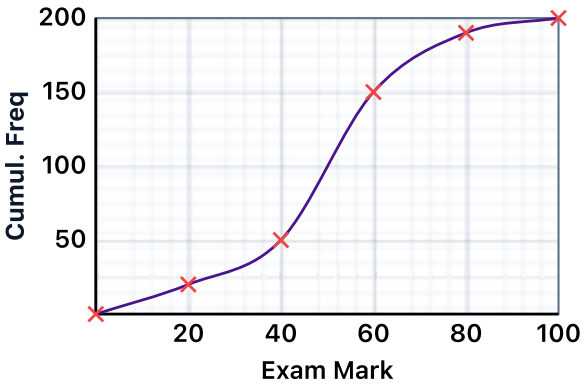
> **Common Mistake:** Reading across from x to the cumulative frequency axis gives how many values are $\leq x$ — for "more than x", always subtract that reading from the total frequency N.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

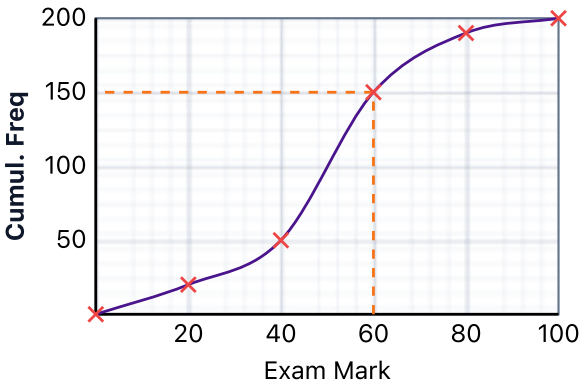
Non-Calculator

The cumulative frequency graph shows the exam marks of 200 candidates.



The pass mark is 60. Estimate how many candidates passed.

STEP-BY-STEP SOLUTION:



Step 1 – Draw a line from $x = 60$ (the threshold) UP to the curve and ACROSS to the y-axis. The reading is 150.

Step 2 – This means 150 candidates scored **60 or less** (since the graph adds up from zero).

Step 3 – To find how many candidates passed (scored *more* than 60), subtract from the total number of candidates (200):

$$200 - 150 = 50$$

Answer: 50 candidates passed.

STATISTICS — AVERAGES & SPREAD

Box Plots

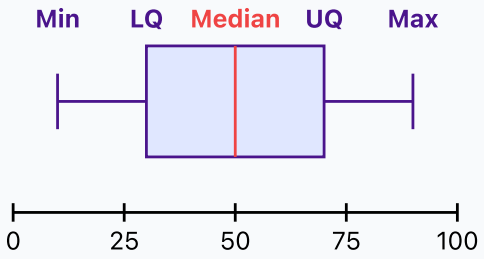
1 The 5-Figure Summary

CORE CONCEPT & METHOD

To draw a box plot, you need 5 numbers:

1. **Minimum**: The lowest value
2. **Lower Quartile (LQ)**: The median of the lower half
3. **Median**: The middle value
4. **Upper Quartile (UQ)**: The median of the upper half
5. **Maximum**: The highest value

Make sure the data is in order first!



> **Common Mistake:** Order all raw values from smallest to largest FIRST before finding the median and quartiles, and end the whiskers at the minimum and maximum data values (never at 0).

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

Find the 5-figure summary for the following data:
4, 9, 11, 15, 18, 22, 24, 28, 31

STEP-BY-STEP SOLUTION:

Step 1 — Data is already ordered. ($n = 9$ values)

Step 2 — Minimum = 4, Maximum = 31

Step 3 — Median: Position $= \frac{9+1}{2} = 5$th value → **18**

Step 4 — Lower Quartile (LQ): Median of lower half {4, 9, 11, 15} → middle of 9 and 11: $\frac{9+11}{2} = \mathbf{10}$

Step 5 — Upper Quartile (UQ): Median of upper half {22, 24, 28, 31} → $\frac{24+28}{2} = \mathbf{26}$

Summary: Min = 4, LQ = 10, Median = 18, UQ = 26, Max = 31

2 Comparing Box Plots

CORE CONCEPT & METHOD

When asked to compare two groups:

1. Compare the **Average** by looking at the **Median** (the line inside the box).
2. Compare the **Consistency/Spread** by looking at the **Interquartile Range** (the width of the box). A smaller IQR means the data is MORE consistent.

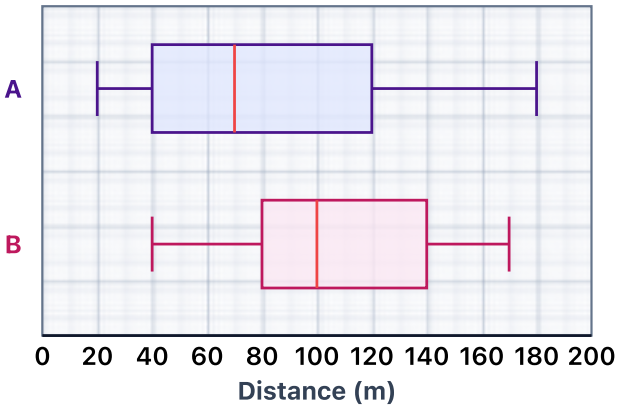
> **Common Mistake:** Every 2-mark distribution comparison MUST compare ONE average (median) AND ONE measure of spread (IQR or range) interpreted in context (e.g. "on average heavier" and "more consistent").

MODEL WORKED EXAMPLES

EXAMPLE 2.1

Non-Calculator

The box plots show the distances (in metres) thrown by athletes in Club A and Club B.



Compare the distances thrown by the two clubs.

STEP-BY-STEP SOLUTION:

1. Compare the Averages (Medians):
Club B threw further on average because their median (100 m) is greater than Club A's median (70 m).

2. Compare the Spread (Interquartile Range):
Club B's distances were more consistent because their IQR ($140 - 80 = 60$ m) is smaller than Club A's IQR ($120 - 40 = 80$ m).

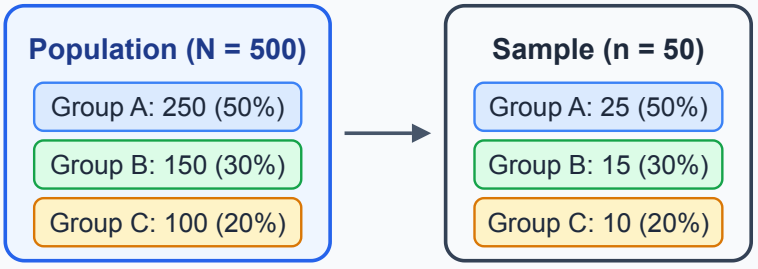
STATISTICS — SAMPLING

Stratified Sampling

1 Calculating Stratified Sample Sizes

CORE CONCEPT & METHOD

A **stratified sample** divides a population into distinct subgroups (called *strata*, such as age groups, departments, or year groups) so that every subgroup is represented in the **exact same proportion** as in the whole population.



The Stratified Sampling Formula:

$$\text{Number sampled from group} = \frac{\text{Group size}}{\text{Total population}} \times \text{Total sample size}$$

- **Step 1: Find the Total Population.** If the total is not stated, add all the group sizes in the table together first.
- **Step 2: Multiply the Fraction by the Sample Size.** Divide the group size by the total population and multiply by the desired sample size.
- **Step 3: Round to the Nearest Whole Number.** Since you cannot sample a fraction of a person, round decimal answers to the nearest whole number (unless exact).
- **Step 4: Selecting the Sample Randomly.** Once you know *how many* to pick from a group, assign a unique number to every member in that group and use a **random number generator** (or draw numbers from a hat) to select the individuals.

Common Mistake: Use $\frac{\text{group size}}{\text{total population}} \times \text{sample size}$ (summing all groups first if no total is shown), and round final sample counts to the nearest whole number.

MODEL WORKED EXAMPLES

EXAMPLE 1.1 Calculator

The table shows the number of students in each house at Oakridge Academy.

House	Falcon	Kestrel	Merlin	Osprey	Total
Students	180	240	150	230	800

A stratified sample of 80 students is taken.
Work out the number of students from **Kestrel** house in the sample.

STEP-BY-STEP SOLUTION:

Step 1: Identify the group size, total population, and sample size

Kestrel group size $= 240$, Total population $= 800$, Sample size $= 80$.

Step 2: Apply the stratified sampling formula

$$\text{Number from Kestrel} = \frac{240}{800} \times 80 = 24$$

$$24$$

EXAMPLE 1.2 Calculator

A leisure centre records the age group of its members in the table below.

Age Group	Under 25	25 to 49	50 and over
Members	140	210	150

The manager takes a stratified sample of 60 members.
Find how many members aged **50 and over** should be in the sample.

STEP-BY-STEP SOLUTION:

Step 1: Add the groups to find the total population

$$\text{Total members} = 140 + 210 + 150 = 500$$

Step 2: Calculate the stratified sample for the '50 and over' group

$$\text{Number sampled} = \frac{150}{500} \times 60 = 18$$

$$18$$

EXAMPLE 1.3

Calculator

The table shows the number of employees working in four departments of a company.

Department	Sales	Design	Finance	Logistics
Employees	128	94	63	175

A stratified sample of 70 employees is selected.
Work out the number of employees from **Design** in the sample.

STEP-BY-STEP SOLUTION:

Step 1: Find the total number of employees

$$\text{Total} = 128 + 94 + 63 + 175 = 460$$

Step 2: Calculate the exact value for Design

$$\frac{94}{460} \times 70 = 14.304...$$

Step 3: Round to the nearest whole number

Rounding $14.304...$ to the nearest integer gives 14.

$$14$$

2 Reverse Stratified Sampling & Missing Table Values

CORE CONCEPT & METHOD

In reverse stratified sampling problems, you are told how many individuals were sampled from **one specific group** instead of being given the total sample size directly.

Key Principle — Constant Sampling Fraction:

In any stratified sample, the ratio of sample size to population size is **identical for every group**:

$$\frac{\text{Sample in Group A}}{\text{Population in Group A}} = \frac{\text{Sample in Group B}}{\text{Population in Group B}} = \frac{\text{Total Sample}}{\text{Total Population}}$$

- **Method 1 (Sampling Fraction):** Find the fraction of the known group that was sampled, $\frac{\text{Group sample}}{\text{Group population}}$, and multiply any other group's population by that same fraction.
- **Method 2 (Scale Factor / Multiplier):** Find how many people in the population correspond to 1 person in the sample by dividing $\text{Group population} \div \text{Group sample}$.
- **Two-Way Tables:** When a population is stratified by both *gender* and *year group* (e.g. Year 10 Girls), use the specific cell value divided by the grand total of the whole two-way table.

Common Mistake: If s people are sampled from a group of g, the sampling fraction is $\frac{s}{g}$ — divide s by $\frac{g}{N}$ (i.e. compute $s \times \frac{N}{g}$) to find the total sample size.

MODEL WORKED EXAMPLES

EXAMPLE 2.1 Calculator

The table shows the number of students in four year groups at a secondary school.

Year Group	Year 7	Year 8	Year 9	Year 10
Number of Students	160	200	240	180

A stratified sample is taken by year group. There are 24 Year 7 students in the sample.
Work out how many **Year 9** students are in the sample.

STEP-BY-STEP SOLUTION:

Step 1: Find the sampling fraction using Year 7

$$\text{Sampling fraction} = \frac{24}{160} = \frac{3}{20}$$

Step 2: Multiply the Year 9 population by the same fraction

$$\text{Year 9 sample} = \frac{3}{20} \times 240 = 36$$

$$36$$

EXAMPLE 2.2 Calculator

The table shows information about the population and a stratified sample of trees in a nature reserve.

Tree Species	Oak	Birch	Ash	Pine
Population	280	350	x	140
Sample	16	20	12	8

Work out the missing population value x for Ash trees.

STEP-BY-STEP SOLUTION:

Step 1: Find the multiplier from Sample to Population using Oak (or Birch)

$$\frac{280}{16} = 17.5$$

Each 1 tree in the sample represents 17.5 trees in the population.

Step 2: Multiply the Ash sample size by 17.5

$$x = 12 \times 17.5 = 210$$

$$210$$

EXAMPLE 2.3

Calculator

The two-way table shows the number of male and female runners in three club categories.

Category	Junior	Senior	Veteran	Total
Female	75	135	60	270
Male	90	150	90	330
Total	165	285	150	600

A sample of 80 runners is stratified by both category and gender.
Work out the number of **Female Senior** runners in the sample.

STEP-BY-STEP SOLUTION:

Step 1: Identify the cell population and grand total

Number of Female Senior runners $= 135$.
Grand total of all runners $= 600$.

Step 2: Calculate the stratified sample size

$$\text{Female Seniors sampled} = \frac{135}{600} \times 80 = 18$$

$$18$$

3 Stratified Sampling with Probability & Ratios

CORE CONCEPT & METHOD

In Higher GCSE exams, stratified sampling is frequently combined with **ratio** or **probability without replacement**.

- **Stratified Sampling with Ratios:** If the population is split into strata in the ratio $a : b : c$, then the fraction belonging to the first group is $\frac{a}{a+b+c}$. Multiply this fraction directly by the sample size n.
- **Probability from a Stratified Sample:** Once a stratified sample of size n has been selected, if **two individuals** are chosen at random from the sample (without replacement), the probability that both belong to a group with k members in the sample is:

$$P(\text{both from group}) = \frac{k}{n} \times \frac{k-1}{n-1}$$

- **Both of the Same Type:** If the sample contains a items of Type A, b of Type B, and c of Type C, the probability that two randomly chosen items from the sample are of the *same type* is:

$$\frac{a(a-1) + b(b-1) + c(c-1)}{n(n-1)}$$

Common Mistake: When picking two items from a stratified sample WITHOUT replacement, subtract 1 from BOTH the numerator and denominator on the second pick (e.g. $\frac{8}{20} \times \frac{7}{19}$).

MODEL WORKED EXAMPLES

EXAMPLE 3.1 Calculator

A box contains 300 festival wristbands of three colours, as shown below.

Colour	Gold	Silver	Bronze	Total
Wristbands	60	105	135	300

A stratified sample of 20 wristbands is selected. Two wristbands are then picked at random from the sample of 20 without replacement.
Work out the probability that **both** wristbands picked from the sample are **Silver**. Give your answer as a simplified fraction.

STEP-BY-STEP SOLUTION:

Step 1: Find how many Silver wristbands are in the sample of 20

$$\text{Silver in sample} = \frac{105}{300} \times 20 = 7$$

Step 2: Find the probability of picking two Silver wristbands without replacement

$$P(\text{both Silver}) = \frac{7}{20} \times \frac{6}{19} = \frac{42}{380} = \frac{21}{190}$$

$$\frac{21}{190}$$

EXAMPLE 3.2 Non-Calculator

At a music college, students study Strings, Woodwind, or Brass in the ratio $5 : 7 : 3$.

Section	Strings	Woodwind	Brass
Ratio Share	5	7	3

A stratified sample of 90 students is selected.
Work out how many **Woodwind** students are in the sample.

STEP-BY-STEP SOLUTION:

Step 1: Find the total number of ratio parts

$$5 + 7 + 3 = 15 \text{ parts}$$

Step 2: Multiply the Woodwind fraction of the population by the sample size

Since the sample is stratified, the sample shares the exact same ratio as the population:

$$\text{Woodwind in sample} = \frac{7}{15} \times 90 = 42$$

$$42$$

EXAMPLE 3.3

Calculator

A crate holds 400 ceramic tiles of three sizes, as shown in the table.

Size	Small	Medium	Large	Total
Tiles	80	200	120	400

A stratified sample of 10 tiles is selected. Two tiles are then chosen at random from the sample of 10 without replacement.
Work out the probability that the two tiles chosen are of the **same size**. Give your answer as a simplified fraction.

STEP-BY-STEP SOLUTION:

Step 1: Find the number of each tile size in the stratified sample of 10

$$\text{Small} = \frac{80}{400} \times 10 = 2$$

$$\text{Medium} = \frac{200}{400} \times 10 = 5$$

$$\text{Large} = \frac{120}{400} \times 10 = 3$$

Step 2: Calculate the probability that both tiles are the same size

$$P(\text{both Small}) = \frac{2}{10} \times \frac{1}{9} = \frac{2}{90}$$

$$P(\text{both Medium}) = \frac{5}{10} \times \frac{4}{9} = \frac{20}{90}$$

$$P(\text{both Large}) = \frac{3}{10} \times \frac{2}{9} = \frac{6}{90}$$

$$P(\text{same size}) = \frac{2 + 20 + 6}{90} = \frac{28}{90} = \frac{14}{45}$$

$$\frac{14}{45}$$

STATISTICS — SAMPLING

Capture-Recapture

1 Estimating Populations

CORE CONCEPT & METHOD

The **Capture-Recapture** method assumes that the fraction of marked items in your second sample is the SAME as the fraction of marked items in the whole population.

$$\frac{\text{Marked in Sample 2}}{\text{Total in Sample 2}} = \frac{\text{Total Marked in Population}}{\text{Total Population}}$$

Cross-multiply and divide to solve!

Common Mistake: Equate tagged proportions $\frac{M}{N} = \frac{m}{n} \implies N = \frac{M \times n}{m}$: multiply the first and second sample sizes, then divide by the number of recaptured marked individuals m.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Calculator

An ecologist catches 36 trout from a reservoir, tags them, and returns them. The next day she catches 45 trout and finds that 6 of them are tagged. Estimate the total number of trout in the reservoir.

STEP-BY-STEP SOLUTION:

Step 1 — Set up the equation:

$$\frac{\text{Marked in sample 2}}{\text{Total in sample 2}} = \frac{\text{Total marked}}{\text{Total population}}$$

$$\frac{6}{45} = \frac{36}{\mathbf{N}}$$

(× 6 above the left fraction; × 6 below the left fraction)

Step 2 — Use equivalent fractions:

Since $6 \times 6 = 36$, we must multiply the bottom by 6 as well to keep the fraction equivalent.

$N = 45 \times 6 = \mathbf{270}$

Answer: The estimated population is **270 trout**.

2 Key Assumptions

CORE CONCEPT & METHOD

For the capture-recapture method to be accurate, we must assume:

- The marks do not wash/rub off.
- The marked items mix completely randomly back into the population.
- The population size doesn't change (no births, deaths, or migration) between the two samples.

Common Mistake: If tags fall off between samples, fewer marked animals m are recaptured, so dividing $\frac{M \times n}{m}$ by a smaller denominator produces an OVERESTIMATE of the population.

MODEL WORKED EXAMPLES

EXAMPLE 2.1 — Non-Calculator

State one mathematical assumption made when estimating the number of fish in a lake using capture-recapture.

STEP-BY-STEP SOLUTION:

Any one of the three standard GCSE assumptions is accepted:

- The marked fish mix completely and randomly back into the population.
- No tags or marks wash off between the two samples.
- The population size remains constant (no births, deaths, or migration).

EXAMPLE 2.2 — Non-Calculator

If some of the paint marks washed off before the second sample was taken, how would this affect the population estimate?

STEP-BY-STEP SOLUTION:

Step 1: If marks washed off, fewer marked fish would be counted in the second sample (making the denominator of $N = \frac{M \times n}{m}$ smaller).

Step 2: Dividing by a smaller number produces a larger result, so the estimate would be an **overestimate** (too high).