

GCSE MATHEMATICS (9-1) — TUTORIALS & WORKED EXAMPLES

Grade 7 Mathematics — Tutorials & Examples

(Higher Tier)

Higher Tier Tutorials, Common Mistake Warnings & Step-by-Step Worked Examples

- **Higher Tier Core Concept Notes** & Precision Geometric/Graph Diagrams
- **Exam-Focused "Common Mistake" Warnings** on every single step
- **Model Worked Examples** tagged **Calculator** or **Non-Calculator**
- **Streamlined Higher Revision Guide** (Tutorials & Worked Examples Only)

Student Name: _____

Class / Teacher: _____

Target Grade: _____



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Simplifying Surds

1 Simplifying Surds

□ CORE CONCEPT & METHOD

A **surd** is a square root that doesn't give a nice whole number (like $\sqrt{2}$ or $\sqrt{5}$).

To simplify a surd, find the **largest square number** (4, 9, 16, 25, 36...) that divides into it.

When adding or subtracting surds, treat them like letters! $3x + 2x = 5x$, so $3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}$.

Note on Adding & Subtracting

To add or subtract surds, the square root part **must be exactly the same**. If the numbers inside the roots are different, you cannot add them directly — you must simplify them first so they match!

COMMON MISTAKE: Never add or subtract numbers under separate square roots ($\sqrt{108} - \sqrt{27} \neq \sqrt{81}$) — simplify each surd to a common root like $6\sqrt{3} - 3\sqrt{3}$ first.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✓ Non-Calculator

Simplify $\sqrt{20}$

STEP-BY-STEP SOLUTION:

1. Find a square number that goes into 20. (4 goes into 20).

2. Rewrite: $\sqrt{20} = \sqrt{4 \times 5}$

3. Split it: $\sqrt{4} \times \sqrt{5}$

4. Square root the 4: $2 \times \sqrt{5}$

Answer: $2\sqrt{5}$

$$2\sqrt{5}$$

EXAMPLE 1.2 **Non-Calculator**Simplify $4\sqrt{5} + 3\sqrt{5}$ **STEP-BY-STEP SOLUTION:**

The surds match, so just add the numbers outside:

$$4 + 3 = 7$$

Answer: $7\sqrt{5}$

$$7\sqrt{5}$$

EXAMPLE 1.3 **Non-Calculator**Simplify $\sqrt{8} + \sqrt{32}$ **STEP-BY-STEP SOLUTION:**

The surds don't match! Simplify them first.

$$1. \sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$$

$$2. \sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$$

$$3. \text{ Now add: } 2\sqrt{2} + 4\sqrt{2} = 6\sqrt{2}$$

Answer: $6\sqrt{2}$

$$6\sqrt{2}$$

2 Dividing Surds **CORE CONCEPT & METHOD**When **dividing** surds, divide the numbers inside the roots under a single square root:

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \text{ (or } \sqrt{a} \div \sqrt{b} = \sqrt{a \div b} \text{)}.$$

When there are whole numbers in front of the surds, divide the **outside numbers** byeach other and divide the **inside numbers** separately: $\frac{c\sqrt{a}}{d\sqrt{b}} = \left(\frac{c}{d}\right)\sqrt{\frac{a}{b}}.$  **Always check whether the resulting surd can be simplified further by factoring out a square number!**

COMMON MISTAKE: You can divide numbers **INSIDE** roots ($\frac{\sqrt{75}}{\sqrt{3}} = \sqrt{25} = 5$), but **NEVER** divide an outside integer by a number inside a root.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 Non-Calculator

Simplify $\frac{\sqrt{72}}{\sqrt{2}}$

STEP-BY-STEP SOLUTION:

1. Combine into a single square root and divide the numbers inside:

$$\sqrt{\frac{72}{2}} = \sqrt{36}$$

2. Evaluate the square root:

$$\sqrt{36} = 6$$

Answer: 6

6

EXAMPLE 2.2

 Non-Calculator

Simplify $\frac{\sqrt{90}}{\sqrt{5}}$

STEP-BY-STEP SOLUTION:

1. Divide the numbers inside the roots:

$$\sqrt{\frac{90}{5}} = \sqrt{18}$$

2. Simplify $\sqrt{18}$ using the largest square factor (9):

$$\sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$$

Answer: $3\sqrt{2}$

 $3\sqrt{2}$

EXAMPLE 2.3

 Non-Calculator

Simplify $\frac{12\sqrt{40}}{3\sqrt{2}}$

STEP-BY-STEP SOLUTION:

1. Divide the numbers outside the roots: $12 \div 3 = 4$

2. Divide the numbers inside the roots: $\sqrt{40 \div 2} = \sqrt{20}$

3. Simplify $\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$

4. Multiply with the number outside: $4 \times 2\sqrt{5} = 8\sqrt{5}$

Answer: $8\sqrt{5}$

 $8\sqrt{5}$

NUMBER — SURDS

Expanding with Surds

CORE CONCEPT & METHOD

When **multiplying** surds, multiply the outside numbers together and multiply the inside numbers together: $c\sqrt{a} \times d\sqrt{b} = cd\sqrt{ab}$.

 **Tip: A surd multiplied by itself gives the number inside!** $\sqrt{a} \times \sqrt{a} = a$

To expand brackets, use the same rules (like FOIL) as you do in Algebra.

COMMON MISTAKE: When squaring a bracket like $(3 + 7\sqrt{2})^2$, write $(3 + 7\sqrt{2})(3 + 7\sqrt{2})$ in full so you do not miss the two middle terms, and square BOTH parts of $(7\sqrt{2})^2 = 49 \times 2 = 98$.

MODEL WORKED EXAMPLES

EXAMPLE 1

 **Non-Calculator**

Expand $\sqrt{5}(\sqrt{2} + \sqrt{5})$

STEP-BY-STEP SOLUTION:

Multiply both terms inside by $\sqrt{5}$:

$$1. \sqrt{5} \times \sqrt{2} = \sqrt{10}$$

$$2. \sqrt{5} \times \sqrt{5} = 5$$

$$\text{Answer: } \sqrt{10} + 5$$

$$\sqrt{10} + 5$$

EXAMPLE 2

 **Non-Calculator**

Expand and simplify $(3 + \sqrt{7})(4 - \sqrt{7})$

STEP-BY-STEP SOLUTION:

Use FOIL (First, Outside, Inside, Last):

$$\text{F: } 3 \times 4 = 12$$

$$\text{O: } 3 \times -\sqrt{7} = -3\sqrt{7}$$

$$\text{I: } \sqrt{7} \times 4 = 4\sqrt{7}$$

$$\text{L: } \sqrt{7} \times -\sqrt{7} = -7$$

$$\text{Collect numbers: } 12 - 7 = 5$$

$$\text{Collect surds: } 4\sqrt{7} - 3\sqrt{7} = \sqrt{7}$$

$$\text{Answer: } 5 + \sqrt{7}$$

$$5 + \sqrt{7}$$

EXAMPLE 3 **Non-Calculator**

Expand and simplify $(5 + \sqrt{3})(5 - \sqrt{3})$

STEP-BY-STEP SOLUTION:

This is a 'Difference of Two Squares'!

F: $5 \times 5 = 25$

O/I: $-5\sqrt{3} + 5\sqrt{3} = 0$

L: $\sqrt{3} \times -\sqrt{3} = -3$

Result: $25 - 3 = 22$

Answer: 22

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Rationalising the Denominator

📖 CORE CONCEPT & METHOD

Mathematicians hate leaving a surd (square root) on the bottom (denominator) of a fraction. To **rationalise** it, we multiply the top and bottom by a clever trick to make the bottom a whole number.

- If the bottom is $\sqrt{a}$, multiply top and bottom by $\sqrt{a}$.
- If the bottom is $a + \sqrt{b}$, multiply top and bottom by its **conjugate** $a - \sqrt{b}$. (This creates a difference of two squares!).

COMMON MISTAKE: For a two-term denominator like $3 - \sqrt{2}$, multiply top and bottom by its CONJUGATE $3 + \sqrt{2}$ (flipping only the middle sign), NEVER by $\sqrt{2}$ alone.

MODEL WORKED EXAMPLES

EXAMPLE 1

🚫 Non-Calculator

Rationalise $\frac{12}{\sqrt{3}}$

STEP-BY-STEP SOLUTION:

Multiply top and bottom by $\sqrt{3}$:

$$\frac{12 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{12\sqrt{3}}{3}$$

Simplify the whole numbers: $12 \div 3 = 4$

Answer: $4\sqrt{3}$

$4\sqrt{3}$

EXAMPLE 2 **Non-Calculator**

Rationalise $\frac{15}{4 + \sqrt{3}}$

STEP-BY-STEP SOLUTION:

Multiply top and bottom by the conjugate $(4 - \sqrt{3})$:

Top: $15(4 - \sqrt{3})$

Bottom (Diff of Squares): $(4 + \sqrt{3})(4 - \sqrt{3}) = 4^2 - (\sqrt{3})^2$
 $= 16 - 3 = 13$

Answer: $\frac{15(4 - \sqrt{3})}{13}$

$$\frac{15(4 - \sqrt{3})}{13}$$

NUMBER — APPROXIMATION & BOUNDS

Calculating with Bounds

1 Bounds of Shapes

□ CORE CONCEPT & METHOD

When a measurement has been rounded, it has an **Upper Bound** and a **Lower Bound**.

To find the maximum possible Area of a rectangle, you must multiply the Upper Bound of the length by the Upper Bound of the width ($UB \times UB$).

COMMON MISTAKE: Halve the rounding unit to find the error interval (± 0.5 for nearest integer, ± 0.05 for 1 d.p.), and use the exact unrounded upper bound $x + \text{half-unit}$ (never $x + 0.49$).

MODEL WORKED EXAMPLES

EXAMPLE 1.1



A rectangular garden has length 15 m and width 9 m, both measured to the nearest metre. Calculate the upper bound for its area.

STEP-BY-STEP SOLUTION:

Length upper bound = 15.5

Width upper bound = 9.5

Area Upper Bound = $15.5 \times 9.5 = 147.25 \text{ m}^2$

2 Bounds of Formulas

□ CORE CONCEPT & METHOD

Division is tricky! To find the maximum possible result of a division ($v = \frac{s}{t}$):

You must divide the LARGEST possible numerator by the SMALLEST possible denominator.

$$\text{Max } v = UB(s) \div LB(t)$$

COMMON MISTAKE: For a division $d = \frac{m}{v}$, the UPPER bound uses $\frac{UB(m)}{LB(v)}$ and the LOWER bound uses $\frac{LB(m)}{UB(v)}$ — never divide upper by upper.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 **Calculator**

Given $v = \frac{d}{t}$, where $d = 160$ (nearest 10) and $t = 8$ (nearest integer). Find the upper bound of v .

STEP-BY-STEP SOLUTION:

To maximize a fraction, divide the upper bound of the numerator by the lower bound of the denominator.

$$d_{\text{upper}} = 165$$

$$t_{\text{lower}} = 7.5$$

$$v_{\text{upper}} = \frac{165}{7.5} = 22$$

3 Suitable Degree of Accuracy

 CORE CONCEPT & METHOD

If a question asks for a 'suitable degree of accuracy', you must:

1. Calculate the Upper Bound of the final answer.
2. Calculate the Lower Bound of the final answer.
3. Find the most decimal places or significant figures that **match** between both numbers.

COMMON MISTAKE: You MUST calculate both UB and LB in full, then round to the most accurate place value where BOTH bounds give the exact same rounded figure.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 **Non-Calculator**

A calculation gives an upper bound of 7.843 and a lower bound of 7.836. What is a suitable degree of accuracy for the answer?

STEP-BY-STEP SOLUTION:

Compare the upper and lower bounds rounded to increasing degrees of accuracy:

Bound	1 d.p.	2 d.p.	3 d.p.
UB = 7.843	7.8	7.84	7.843
LB = 7.836	7.8	7.84	7.836
Comparison	Same	Same ← Choose	Different

Both bounds agree to 1 d.p. and 2 d.p., but differ at 3 d.p.. We choose the most accurate level where both bounds still match: 7.84 (to 2 d.p.).

Rearranging Harder Formulae

CORE CONCEPT & METHOD

When the letter you want to make the subject appears more than once, you need to collect all of them on one side and then **factorise!**

COMMON MISTAKE: Whenever the target letter x appears TWICE, collect every x -term on one side and factor x out as a single bracket — your final expression must NEVER still contain x on the right-hand side.

MODEL WORKED EXAMPLES

EXAMPLE 1

 Non-Calculator

Make x the subject of the formula: $px = qx + r$

STEP-BY-STEP SOLUTION:

1. Get all the x 's on one side! Subtract qx from both sides:

$$\begin{array}{rcl} px & = & qx + r \\ -qx & & -qx \\ \hline px - qx & = & r \end{array}$$

2. The letter x is trapped in two terms. Factorise it out!

$$x(p - q) = r$$

3. Divide by the bracket to get x on its own:

$$\begin{array}{rcl} x(p - q) & = & r \\ \div (p - q) & & \div (p - q) \\ \hline x & = & \frac{r}{p - q} \end{array}$$

$$x = \frac{r}{p - q}$$

EXAMPLE 2 **Non-Calculator**

Make x the subject of the formula: $y = \frac{x+4}{x-5}$

STEP-BY-STEP SOLUTION:

1. Multiply both sides by $(x-5)$ to clear the fraction:

$$\begin{array}{rcl} y & = & \frac{x+4}{x-5} \\ \times(x-5) & & \times(x-5) \\ \hline y(x-5) & = & x+4 \end{array}$$

2. Expand the bracket:

$$yx - 5y = x + 4$$

3. Get all the x 's on the left, and everything else on the right. Subtract x :

$$\begin{array}{rcl} yx - 5y & = & x + 4 \\ -x & & -x \\ \hline yx - x - 5y & = & 4 \end{array}$$

4. Add $5y$ to the right:

$$\begin{array}{rcl} yx - x - 5y & = & 4 \\ +5y & & +5y \\ \hline yx - x & = & 4 + 5y \end{array}$$

5. Factorise x out on the left:

$$x(y-1) = 4 + 5y$$

6. Divide by the bracket:

$$\begin{array}{rcl} x(y-1) & = & 4 + 5y \\ \div(y-1) & & \div(y-1) \\ \hline x & = & \frac{4 + 5y}{y-1} \end{array}$$

$$x = \frac{4 + 5y}{y - 1}$$

EXAMPLE 3 **Non-Calculator**

Make x the subject of the formula: $y = \frac{3x + 5}{2x - 7}$

STEP-BY-STEP SOLUTION:

1. Multiply both sides by $(2x - 7)$ to clear the fraction:

$$y(2x - 7) = 3x + 5$$

2. Expand the bracket:

$$2xy - 7y = 3x + 5$$

3. Collect all x terms on the left and non- x terms on the right (subtract $3x$ and add $7y$):

$$2xy - 3x = 7y + 5$$

4. Factorise x out on the left:

$$x(2y - 3) = 7y + 5$$

5. Divide both sides by $(2y - 3)$:

$$x = \frac{7y + 5}{2y - 3}$$

$$x = \frac{7y + 5}{2y - 3}$$

ALGEBRA — QUADRATICS

Factorising Harder Quadratics

□ CORE CONCEPT & METHOD

When the x^2 has a number in front (like $2x^2$), the puzzle changes. Follow these steps:

1. **Multiply** the first number by the last number.
2. **Find factors** of *that* new number which add to the middle term.
3. **Split** the middle x term into those two pieces.
4. **Factorise** each half!

COMMON MISTAKE: In $ax^2 + bx + c$ with $a > 1$, find two numbers that multiply to $a \times c$ (NOT c alone) and add to b , then split the middle bx term before factorising in pairs.

MODEL WORKED EXAMPLES

EXAMPLE 1

Factorise $2x^2 + 9x + 4$

✎ Non-Calculator

STEP-BY-STEP SOLUTION:

1. Multiply first and last: $2 \times 4 = 8$.
2. Find two numbers that **Multiply** to $+8$ and **Add** to $+9$.

Multiply = $+8$	Add = $+9$
1×8	$1 + 8 = +9$ ✓
2×4	$2 + 4 = 6$ ✗

3. Split the middle term using $+8x$ and $+1x$:

$$2x^2 + 8x + 1x + 4$$

4. Factorise the first half ($2x^2 + 8x$):
 $\rightarrow 2x(x + 4)$

5. Factorise the second half ($+1x + 4$):
 $\rightarrow +1(x + 4)$

The brackets match! Pull them together:

$$(2x + 1)(x + 4)$$

EXAMPLE 2 **Non-Calculator****Factorise** $3x^2 - 5x - 2$ **STEP-BY-STEP SOLUTION:**

1. Multiply first and last: $3 \times -2 = -6$.
2. Find two numbers that **Multiply** to -6 and **Add** to -5 .

Multiply = -6	Add = -5
-2×3	$-2 + 3 = +1$ ✗
-6×1	$-6 + 1 = -5$ ✓

3. Split the middle term using $-6x$ and $+1x$:

$$3x^2 - 6x + 1x - 2$$

4. Factorise the first half ($3x^2 - 6x$):
 $\rightarrow 3x(x - 2)$

5. Factorise the second half ($+1x - 2$):
 $\rightarrow +1(x - 2)$

The brackets match! Pull them together:

$$(3x + 1)(x - 2)$$

The Quadratic Formula

1 Solving Harder Quadratics ($a > 1$)

□ CORE CONCEPT & METHOD

When there is a number in front of the x^2 (like $2x^2 + 5x + 3 = 0$), we use the **AC Method** to split the middle term!

1. Multiply the first number (a) by the last number (c).
2. Find two numbers that multiply to make this new number, but still add up to the middle number (b).
3. Split the middle x term into those two numbers.
4. Factorise the first half and the second half separately to get your double brackets!

COMMON MISTAKE: After factorising $(2x + 3)(x - 4) = 0$, set each bracket equal to zero to flip the signs and divide by the x -coefficient ($x = -\frac{3}{2}$, NOT $+3$ or -3).

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✎ Non-Calculator

Solve $2x^2 + 9x + 4 = 0$

STEP-BY-STEP SOLUTION:

1. Multiply $a \times c$: $2 \times 4 = 8$.
2. Numbers that multiply to 8 and add to 9: 8 and 1.
3. Split the middle term: $2x^2 + 8x + 1x + 4 = 0$
4. Factorise first half: $2x(x + 4)$
5. Factorise second half: $+1(x + 4)$
6. Brackets: $(2x + 1)(x + 4) = 0$
7. Set each bracket to zero:

$$2x + 1 = 0 \implies x = -0.5$$

$$x + 4 = 0 \implies x = -4$$

$$x = -0.5, x = -4$$

EXAMPLE 1.2 **Non-Calculator**

Solve $6x^2 - 5x - 4 = 0$

STEP-BY-STEP SOLUTION:

1. Multiply $a \times c$: $6 \times -4 = -24$.
2. Numbers that multiply to -24 and add to -5 : -8 and 3 .
3. Split the middle term: $6x^2 - 8x + 3x - 4 = 0$
4. Factorise first half: $2x(3x - 4)$
5. Factorise second half: $+1(3x - 4)$
6. Brackets: $(2x + 1)(3x - 4) = 0$
7. Set each bracket to zero:
 $2x + 1 = 0 \implies x = -0.5$
 $3x - 4 = 0 \implies x = \frac{4}{3}$

$$x = -0.5, x = \frac{4}{3}$$

2 Quadratic Formula **CORE CONCEPT & METHOD**

When a quadratic won't factorise easily, you can use the Quadratic Formula!

First, ensure your equation is in the standard form:

$$ax^2 + bx + c = 0$$

This tells you exactly what numbers to substitute for a , b , and c in the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Just substitute a , b , and c carefully. Don't forget that if b is negative, $-b$ becomes positive!

COMMON MISTAKE: When b is negative, wrap it in brackets inside b^2 so $(-5)^2 = +25$ (never -25), and watch double negatives in $-4ac$ when $c < 0$.

MODEL WORKED EXAMPLES

EXAMPLE 2.1 **Calculator**

Solve $x^2 + 6x + 4 = 0$ using the quadratic formula. Give your answers to 2 decimal places.

STEP-BY-STEP SOLUTION:

Here, $a = 1$, $b = 6$, and $c = 4$.

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-6 \pm \sqrt{36 - 16}}{2}$$

$$x = \frac{-6 \pm \sqrt{20}}{2}$$

Calculate the two values:

$$x = -0.76, \text{ or } x = -5.24$$

EXAMPLE 2.2 **Calculator**

Solve $2x^2 - 5x - 4 = 0$ using the quadratic formula. Give your answers to 2 decimal places.

STEP-BY-STEP SOLUTION:

Here, $a = 2$, $b = -5$, and $c = -4$.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-4)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{25 - (-32)}}{4}$$

$$x = \frac{5 \pm \sqrt{57}}{4}$$

Calculate the two values:

$$x = -0.64, \text{ or } x = 3.14$$

Quadratic Sequences

□ CORE CONCEPT & METHOD

If the first differences are not the same, but the **second** differences are, it's a quadratic sequence!

Every quadratic sequence follows the standard form:

$$an^2 + bn + c$$

There are two main methods to find the n th term. You can choose whichever you prefer!

Method 1: Subtraction

1. Halve the second difference to find a .
2. Subtract your an^2 sequence from the original sequence to get a normal linear sequence.
3. Find the n th term of that linear sequence to give you the $bn + c$ part!

Example: 4, 11, 22, 37, ...

First differences: 7, 11, 15. Second difference: 4.

Half of 4 is 2, so it starts with $2n^2$.

Original:	4	11	22	37
$2n^2$:	2	8	18	32
Subtract:	2	3	4	5

The linear sequence is 2, 3, 4, 5, which has an n th term of $n + 1$.

Final Answer: $2n^2 + n + 1$

Method 2: Algebraic

Memorise these three equations to find a , b , and c in order:

- $2a$ = second difference
- $3a + b$ = first difference (between 1st & 2nd term)
- $a + b + c$ = first term

Example: 5, 11, 19, 29, 41, ...

$$\begin{array}{rcccccc}
 a + b + c \rightarrow & \mathbf{5} & 11 & 19 & 29 & 41 \\
 3a + b \rightarrow & \mathbf{6} & 8 & 10 & 12 & \\
 2a \rightarrow & \mathbf{2} & 2 & 2 & &
 \end{array}$$

$$2a = 2$$

$$a = 1$$

$$3a + b = 6$$

$$3(1) + b = 6$$

$$3 + b = 6$$

$$b = 3$$

$$a + b + c = 5$$

$$1 + 3 + c = 5$$

$$4 + c = 5$$

$$c = 1$$

The sequence is $an^2 + bn + c$

Final Answer: $n^2 + 3n + 1$

COMMON MISTAKE: The coefficient a of n^2 is HALF the second difference ($2a = 2\text{nd diff}$), NOT the full second difference — always check your formula against $n = 1, 2, 3$.

MODEL WORKED EXAMPLES

EXAMPLE 1

Non-Calculator

Find the n th term of the sequence: 6, 15, 28, 45, ...

STEP-BY-STEP SOLUTION:

	6	15	28	45
	∖∖	∖∖	∖∖	
First Difference	9	13	17	
	∖∖	∖∖		
Second Difference		4	4	

Half the second difference = $4 \div 2 = 2$ So the sequence starts with $2n^2$.

Original:	6	15	28	45
$2n^2$:	2	8	18	32
Subtract:	4	7	10	13

The n th term of 4, 7, 10, 13 is $3n + 1$.

Put them together:

$$2n^2 + 3n + 1$$

EXAMPLE 2

Non-Calculator

Find the n th term of the decreasing sequence: 25, 20, 11, -2, ...

STEP-BY-STEP SOLUTION:

1. Find the first and second differences:

- First differences: -5, -9, -13
- Second difference: -4

2. Halve the second difference: $-4 \div 2 = -2$, so the sequence starts with $-2n^2$.3. Subtract $-2n^2$ from the original sequence:

Original:	25	20	11	-2
$-2n^2$:	-2	-8	-18	-32
Subtract:	27	28	29	30

The linear sequence 27, 28, 29, 30 has n th term $n + 26$.

$$-2n^2 + n + 26$$

Functions

1 Evaluating Functions

□ CORE CONCEPT & METHOD

Instead of writing $y = 3x - 5$, we can write $f(x) = 3x - 5$.

This is called **function notation**.

- **Evaluate $f(4)$:** Substitute $x = 4$ into the function.
 $f(4) = 3(4) - 5 = 7$.
- **Solve $f(x) = 10$:** Set the function equal to 10 and solve for x .
 $3x - 5 = 10 \Rightarrow 3x = 15 \Rightarrow x = 5$.

COMMON MISTAKE: In $f(-2)$ for $f(x) = 3x^2 - 4x$, square $(-2)^2 = +4$ BEFORE multiplying by 3, and treat $-4(-2)$ as $+8$.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✎ Non-Calculator

Given that $f(x) = 6x + 5$, find $f(4)$.

STEP-BY-STEP SOLUTION:

Substitute $x = 4$ into the function.

$$f(4) = 6(4) + 5$$

$$f(4) = 24 + 5 = 29$$

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EXAMPLE 1.2

✎ Non-Calculator

Given that $f(x) = 3x - 8$, solve $f(x) = 19$.

STEP-BY-STEP SOLUTION:

Set the function equal to 19 and solve for x .

$$3x - 8 = 19$$

$$3x = 27$$

$$x = 9$$

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2 Composite Functions

□ CORE CONCEPT & METHOD

A composite function is when you put one function **inside** another.

- $fg(x)$ means $f(g(x))$: You put $g(x)$ inside $f(x)$. Note: You do g FIRST, then f .
- **Evaluating $fg(3)$** : First find $g(3)$. Then put that answer into f .

COMMON MISTAKE: Work from the INSIDE out: $fg(x)$ means $f(g(x))$ — substitute the entire expression for $g(x)$ into f , NEVER multiply $f(x) \times g(x)$.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

✎ Non-Calculator

Given that $f(x) = 4x - 3$ and $g(x) = 2x + 5$, find $fg(x)$.

STEP-BY-STEP SOLUTION:

Substitute $g(x)$ into $f(x)$ wherever there is an x .

$$f(g(x)) = 4(g(x)) - 3$$

$$fg(x) = 4(2x + 5) - 3$$

$$fg(x) = 8x + 20 - 3 = 8x + 17$$

$$8x + 17$$

EXAMPLE 2.2

✎ Non-Calculator

Given that $f(x) = 4x + 5$ and $g(x) = x - 8$, find $fg(6)$.

STEP-BY-STEP SOLUTION:

First, evaluate the inner function $g(6)$.

$$g(6) = (6) - 8 = -2$$

Now substitute this result into f .

$$f(-2) = 4(-2) + 5 = -8 + 5 = -3$$

$$-3$$

EXAMPLE 2.3 **Non-Calculator**

Given that $f(x) = x^2 + 3$ and $g(x) = 2x - 1$, solve the equation $fg(x) = 28$.

STEP-BY-STEP SOLUTION:

1. Substitute $g(x)$ into $f(x)$:

$$fg(x) = (2x - 1)^2 + 3$$

2. Set $fg(x) = 28$ and solve:

$$(2x - 1)^2 + 3 = 28$$

$$(2x - 1)^2 = 25$$

$$2x - 1 = \pm 5$$

$$2x = 6 \implies x = 3 \text{ or } 2x = -4 \implies x = -2$$

$$x = 3, -2$$

3 Inverse Functions **CORE CONCEPT & METHOD**

The inverse function $f^{-1}(x)$ does the exact opposite of $f(x)$. It works backwards!

To find an inverse function algebraically:

1. Write $y = f(x)$.
2. Rearrange the formula to make x the subject.
3. Swap y for x at the very end.

COMMON MISTAKE: $f^{-1}(x)$ means the INVERSE function (rearrange $y = f(x)$ and swap x and y), NOT the reciprocal $\frac{1}{f(x)}$.

MODEL WORKED EXAMPLES**EXAMPLE 3.1** **Non-Calculator**

Given that $f(x) = 5x - 9$, find an expression for $f^{-1}(x)$.

STEP-BY-STEP SOLUTION:

1. Write as $y = 5x - 9$.

2. Rearrange for x :

$$y + 9 = 5x$$

$$\frac{y + 9}{5} = x$$

3. Swap y for x :

$$f^{-1}(x) = \frac{x + 9}{5}$$

$$\frac{x + 9}{5}$$

EXAMPLE 3.2 **Non-Calculator**

Given that $f(x) = \frac{2x + 3}{x - 4}$, find an expression for $f^{-1}(x)$.

STEP-BY-STEP SOLUTION:

1. Write $y = \frac{2x + 3}{x - 4}$ and multiply both sides by $(x - 4)$:

$$y(x - 4) = 2x + 3 \implies xy - 4y = 2x + 3$$

2. Collect x terms on one side and factorise:

$$xy - 2x = 4y + 3 \implies x(y - 2) = 4y + 3 \implies x = \frac{4y + 3}{y - 2}$$

3. Swap y for x :

$$f^{-1}(x) = \frac{4x + 3}{x - 2}$$

$$\frac{4x + 3}{x - 2}$$

Trig & Exponential Graphs

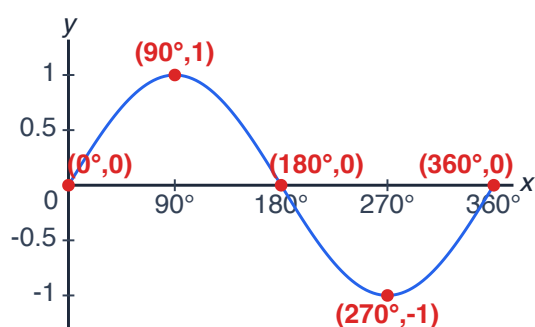
1 Trigonometric Graphs ($y = \sin x$, $y = \cos x$, $y = \tan x$)

□ CORE CONCEPT & METHOD

Exact Trigonometric Graphs ($0^\circ \leq x \leq 360^\circ$)

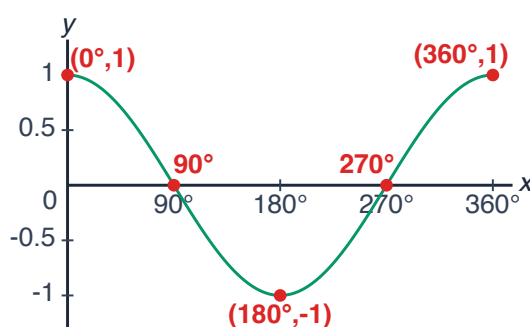
You must be able to recognise, sketch, and state the key coordinates and symmetries of the three trigonometric graphs for $0^\circ \leq x \leq 360^\circ$.

1. The Sine Graph: $y = \sin x^\circ$



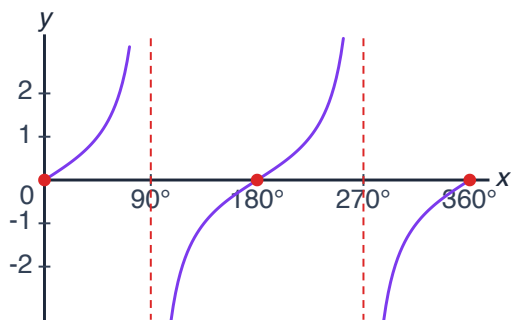
- Starts at $(0^\circ, 0)$.
- Maximum at $(90^\circ, 1)$; minimum at $(270^\circ, -1)$.
- Crosses the x -axis at $0^\circ, 180^\circ, 360^\circ$.
- Period = 360° , range $-1 \leq y \leq 1$.

2. The Cosine Graph: $y = \cos x^\circ$



- Starts at $(0^\circ, 1)$ (bucket/valley shape).
- Minimum at $(180^\circ, -1)$; maxima at $(0^\circ, 1)$ and $(360^\circ, 1)$.
- Crosses the x -axis at 90° and 270° .
- Symmetric about the line $x = 180^\circ$.

3. The Tangent Graph: $y = \tan x^\circ$



- Passes through $(0^\circ, 0)$, $(180^\circ, 0)$, and $(360^\circ, 0)$.
- Has **vertical asymptotes** at $x = 90^\circ$ and $x = 270^\circ$ (where $\tan x$ is undefined).
- Period = 180° , with no maximum or minimum value.

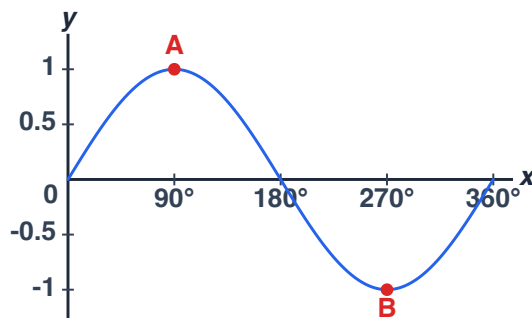
COMMON MISTAKE: $y = \sin x^\circ$ starts at $(0, 0)$, whereas $y = \cos x^\circ$ starts at $(0, 1)$, and $y = \tan x^\circ$ has vertical asymptotes at $x = 90^\circ$ and $x = 270^\circ$.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

Here are sketches of $y = \sin x^\circ$ and $y = \cos x^\circ$ for $0 \leq x \leq 360$.



State the coordinates of the maximum point A and the minimum point B on $y = \sin x^\circ$, and state the equations of the vertical asymptotes of $y = \tan x^\circ$ for $0 \leq x \leq 360$.

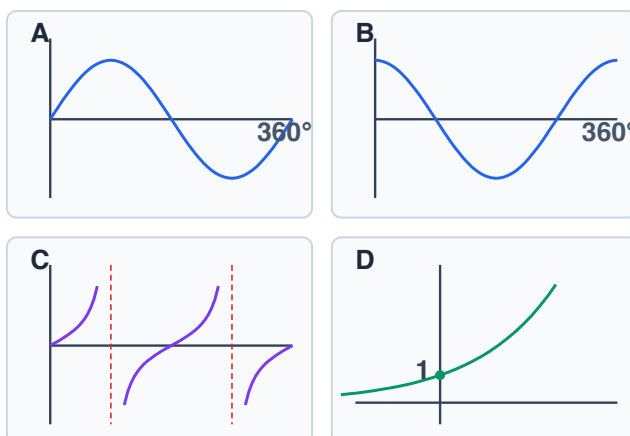
STEP-BY-STEP SOLUTION:

1. On $y = \sin x^\circ$, the curve reaches its maximum value of 1 at $x = 90^\circ$, so $A = (90^\circ, 1)$.
2. It reaches its minimum value of -1 at $x = 270^\circ$, so $B = (270^\circ, -1)$.
3. On $y = \tan x^\circ$, the vertical asymptotes occur at $x = 90^\circ$ and $x = 270^\circ$.

$$A(90^\circ, 1), B(270^\circ, -1), x = 90^\circ, 270^\circ$$

EXAMPLE 1.2 **Non-Calculator**

Match each graph A, B, C, D below to its equation from: $y = \sin x^\circ$, $y = \cos x^\circ$, $y = \tan x^\circ$, and $y = 2^x$.

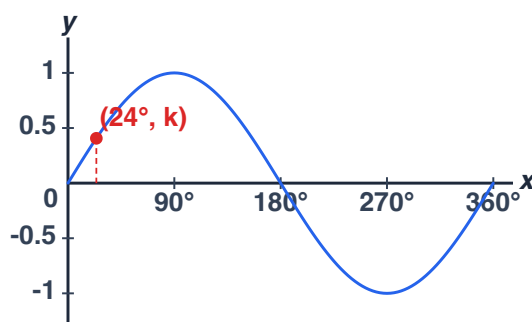
**STEP-BY-STEP SOLUTION:**

- Graph A** starts at $(0^\circ, 0)$, rises to $(90^\circ, 1)$, and completes one wave at $360^\circ \rightarrow y = \sin x^\circ$.
- Graph B** starts at $(0^\circ, 1)$ and has a minimum at $(180^\circ, -1) \rightarrow y = \cos x^\circ$.
- Graph C** has vertical asymptotes at 90° and $270^\circ \rightarrow y = \tan x^\circ$.
- Graph D** is an increasing exponential curve passing through $(0, 1) \rightarrow y = 2^x$.

$A : y = \sin x^\circ$, $B : y = \cos x^\circ$, $C : y = \tan x^\circ$, $D : y = 2^x$

EXAMPLE 1.3 **Non-Calculator**

Here is a sketch of $y = \sin x^\circ$ for $0 \leq x \leq 360$.



Given that $\sin 24^\circ = k$, write down in terms of k the values of $\sin 156^\circ$ and $\sin 204^\circ$.

STEP-BY-STEP SOLUTION:

- By symmetry of the first arch of $y = \sin x^\circ$ about $x = 90^\circ$, since $180^\circ - 24^\circ = 156^\circ$:

$$\sin 156^\circ = \sin 24^\circ = k$$
- Between 180° and 360° , the arch is inverted below the x -axis. Since $180^\circ + 24^\circ = 204^\circ$:

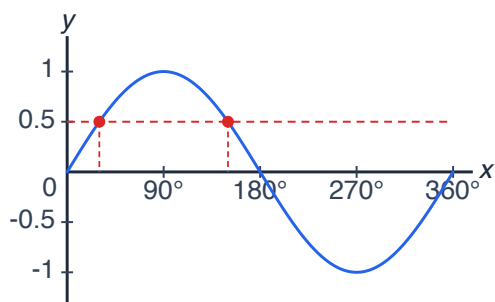
$$\sin 204^\circ = -\sin 24^\circ = -k$$

$$\sin 156^\circ = k, \quad \sin 204^\circ = -k$$

2 Solving Trigonometric Equations & Modelling **CORE CONCEPT & METHOD****Solving Trigonometric Equations & Real-World Models**

Because trigonometric graphs repeat and have symmetry, an equation like $\sin x = 0.5$ has **more than one solution** in the interval $0^\circ \leq x \leq 360^\circ$.

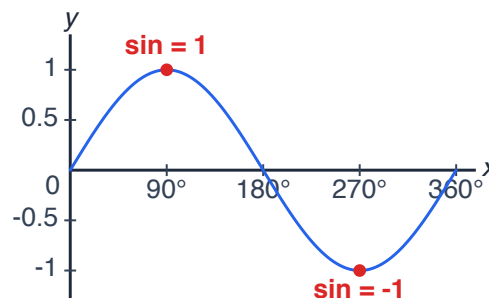
1. Secondary Value Symmetry Rules



Once you find the first solution x_1 , use symmetry to find the second solution in $0^\circ \leq x \leq 360^\circ$:

- **Sine** ($y = \sin x$): $x_2 = 180^\circ - x_1$ (or $540^\circ - x_1$ if negative)
- **Cosine** ($y = \cos x$): $x_2 = 360^\circ - x_1$
- **Tangent** ($y = \tan x$): $x_2 = x_1 + 180^\circ$

2. Trigonometric Modelling ($d = a \pm b \sin(kt)^\circ$)



Since $-1 \leq \sin(\theta) \leq 1$ and $-1 \leq \cos(\theta) \leq 1$:

- **Maximum value** = $a + |b|$
- **Minimum value** = $a - |b|$

Set the inside angle equal to 90° or 270° to find when it occurs!

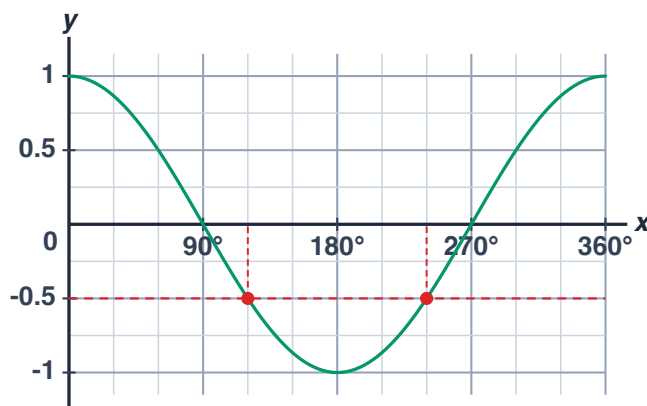
COMMON MISTAKE: Your calculator gives only the principal angle — use symmetry ($180^\circ - \theta$ for sin, $360^\circ - \theta$ for cos, $180^\circ + \theta$ for tan) to find the second root in $0 \leq x \leq 360$.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

Non-Calculator

Here is the graph of $y = \cos x^\circ$ for $0 \leq x \leq 360$.



Use the graph to solve $\cos x^\circ = -0.5$ for $0 \leq x \leq 360$.

STEP-BY-STEP SOLUTION:

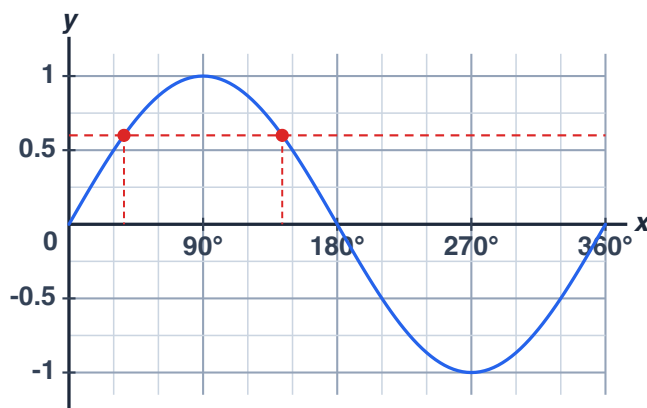
1. Draw the horizontal line $y = -0.5$ across the graph.
2. Read the x -coordinates where $y = -0.5$ intersects $y = \cos x^\circ$: the first intersection is at $x_1 = 120^\circ$.
3. Using symmetry about $x = 180^\circ$, the second solution is $x_2 = 360^\circ - 120^\circ = 240^\circ$.

$$x = 120^\circ, 240^\circ$$

EXAMPLE 2.2

Calculator

Solve $\sin x^\circ = 0.6$ for $0 \leq x \leq 360$, giving your answers correct to 1 decimal place.



STEP-BY-STEP SOLUTION:

1. Find the principal acute angle using inverse sine:

$$x_1 = \sin^{-1}(0.6) = 36.9^\circ \text{ (1 d.p.)}$$

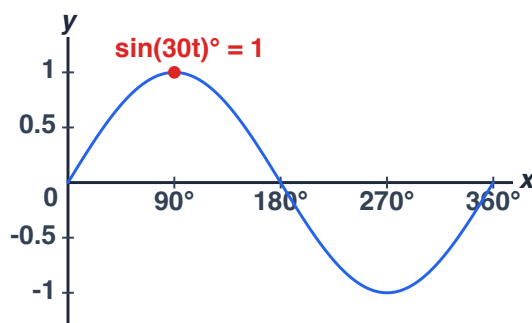
2. For $y = \sin x^\circ$, use symmetry about $x = 90^\circ$ ($x_2 = 180^\circ - x_1$):

$$x_2 = 180^\circ - 36.9^\circ = 143.1^\circ$$

$$x = 36.9^\circ, 143.1^\circ$$

EXAMPLE 2.3**Non-Calculator**

The depth of water, d metres, in a harbour is modelled by $d = 6 - 3 \sin(30t)^\circ$, where t is the number of hours after midnight ($0 \leq t \leq 12$).



Find the minimum depth of water and the time t when it occurs.

STEP-BY-STEP SOLUTION:

1. Since $-1 \leq \sin(30t)^\circ \leq 1$, the depth $d = 6 - 3 \sin(30t)^\circ$ is at its **minimum** when $\sin(30t)^\circ = 1$:

$$d_{\min} = 6 - 3(1) = 3 \text{ m}$$

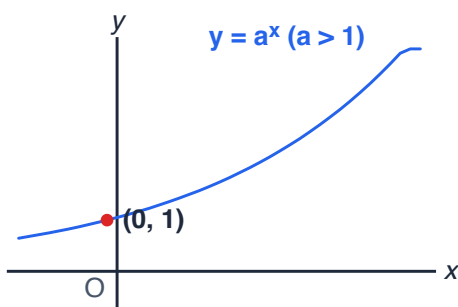
2. From the sine graph, $\sin(30t)^\circ = 1$ when the angle is 90° :

$$30t = 90 \implies t = 3 \text{ hours}$$

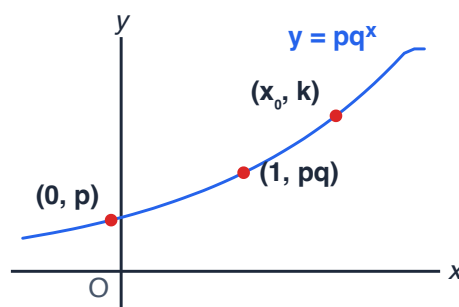
$$d_{\min} = 3 \text{ m at } t = 3$$

3 Exponential Graphs ($y = k^x$ and $y = pq^x$)**CORE CONCEPT & METHOD****Exponential Graphs ($y = k^x$ and $y = pq^x$)**

An exponential function has the variable x in the power (exponent), producing rapid growth or decay.

1. Basic Exponential Curve: $y = a^x$ 

- Because $a^0 = 1$, $y = a^x$ **always crosses the y -axis at $(0, 1)$.**
- The x -axis ($y = 0$) is a **horizontal asymptote** ($a^x > 0$ for all x).
- Growth if $a > 1$; decay if $0 < a < 1$.

2. Finding p , q , and k on $y = pq^x$ 

- Substitute the y -intercept $(0, y_0)$: since $q^0 = 1$, we immediately get $p = y_0$.
- Substitute the second given point into $y = pq^x$ to solve for q .
- Once p and q are known, substitute the x -coordinate of the third point to calculate k !

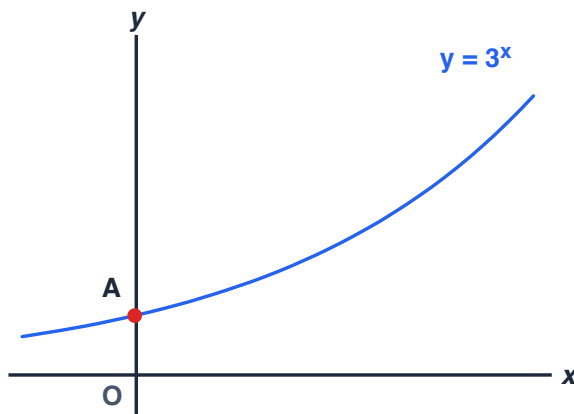
COMMON MISTAKE: In $y = pq^x$, substitute $(0, y_0)$ first to get $p = y_0$ since $q^0 = 1$ (NOT 0), and remember $pq^x = p \times (q^x)$, NEVER $(pq)^x$.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 **Non-Calculator**

The diagram shows a sketch of the curve $y = 3^x$.



Write down the coordinates of the y -intercept A and state the equation of the horizontal asymptote of the curve.

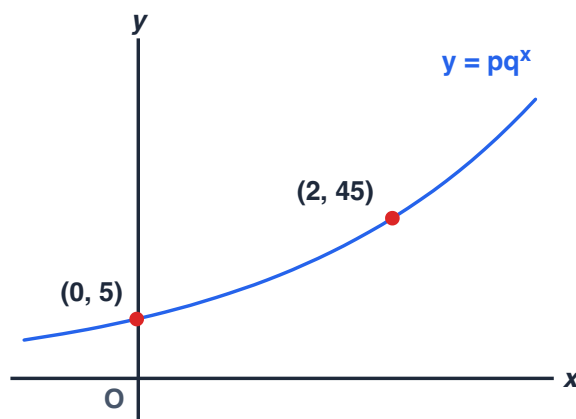
STEP-BY-STEP SOLUTION:

1. On the y -axis, $x = 0$. Since $3^0 = 1$, the y -intercept is $A = (0, 1)$.
2. As $x \rightarrow -\infty$, $3^x \rightarrow 0$ but never reaches 0. Therefore, the horizontal asymptote is the x -axis, which has equation $y = 0$.

$$A(0, 1), \quad y = 0$$

EXAMPLE 3.2 **Non-Calculator**

The diagram shows a sketch of the curve $y = pq^x$, where p and q are positive constants. The curve passes through $(0, 5)$ and $(2, 45)$.



Work out the value of p and the value of q .

STEP-BY-STEP SOLUTION:

1. Substitute $(0, 5)$ into $y = pq^x$:

$$5 = p \times q^0 = p \implies p = 5$$

2. Substitute $(2, 45)$ into $y = 5q^x$:

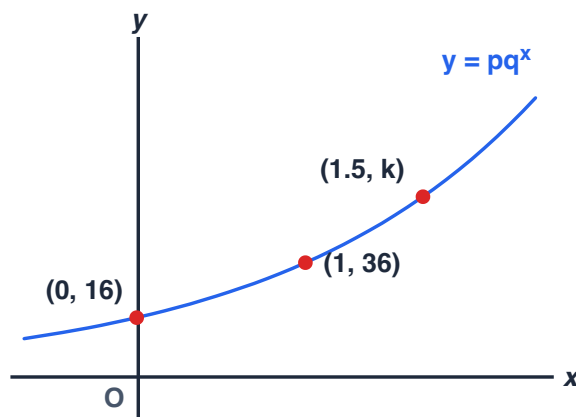
$$45 = 5q^2 \implies q^2 = 9$$

3. Since $q > 0$, $q = 3$.

$$p = 5, \quad q = 3$$

EXAMPLE 3.3 **Non-Calculator**

The diagram shows a sketch of the curve $y = pq^x$ passing through $(0, 16)$, $(1, 36)$, and $(1.5, k)$.



Work out the value of k .

STEP-BY-STEP SOLUTION:

1. Substitute $(0, 16)$ into $y = pq^x$: $p = 16$.

2. Substitute $(1, 36)$ into $y = 16q^x$:

$$36 = 16q \implies q = \frac{36}{16} = \frac{9}{4} = 2.25$$

3. Substitute $x = 1.5$ into $y = 16 \times \left(\frac{9}{4}\right)^x$:

$$k = 16 \times \left(\frac{9}{4}\right)^{1.5} = 16 \times \left(\frac{3}{2}\right)^3 = 16 \times \frac{27}{8} = 54$$

$$k = 54$$

Algebraic Fractions

1 Simplifying Fractions

□ CORE CONCEPT & METHOD

You cannot just cancel out individual terms in an algebraic fraction if they are being added or subtracted!

You **MUST factorise the top and the bottom** first. Only then can you cancel out matching brackets!

COMMON MISTAKE: NEVER cancel individual x^2 or x terms across $+$ or $-$ signs — always factorise numerator and denominator completely first, then cancel matching brackets.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✎ Non-Calculator

Simplify fully $\frac{x^2 + 2x - 15}{x^2 - 9x + 18}$

STEP-BY-STEP SOLUTION:

1. Factorise the top quadratic:

Multiplies to -15, adds to +2. The numbers are -3 and 5.

$$(x - 3)(x + 5)$$

2. Factorise the bottom quadratic:

Multiplies to +18, adds to -9. The numbers are -3 and -6.

$$(x - 3)(x - 6)$$

3. Write them as a fraction and cancel the matching bracket:

$$\frac{(x - 3)(x + 5)}{(x - 3)(x - 6)}$$

$$\frac{x + 5}{x - 6}$$

EXAMPLE 1.2 **Non-Calculator**

Simplify fully $\frac{4x^2 + 20x}{x^2 - 25}$

STEP-BY-STEP SOLUTION:

1. Factorise the top (single bracket!):

Pull out the common factor of $4x$.

$$4x(x + 5)$$

2. Factorise the bottom (Difference of Two Squares!):

$$x^2 - 25 \text{ becomes } (x + 5)(x - 5)$$

3. Write them as a fraction and cancel the matching bracket:

$$\frac{4x(x + 5)}{(x - 5)(x + 5)}$$

$$\frac{4x}{x - 5}$$

EXAMPLE 1.3 **Non-Calculator**

Simplify fully $\frac{2x^2 + 9x - 5}{x^2 + 4x - 5}$

STEP-BY-STEP SOLUTION:

1. Factorise the harder quadratic on top:

$$(2x - 1)(x + 5)$$

2. Factorise the bottom quadratic:

$$(x - 1)(x + 5)$$

3. Write them as a fraction and cancel the matching bracket:

$$\frac{(2x - 1)(x + 5)}{(x - 1)(x + 5)}$$

$$\frac{2x - 1}{x - 1}$$

2 Adding & Subtracting Fractions

📖 CORE CONCEPT & METHOD

To add or subtract algebraic fractions, you need a **common denominator**, just like normal fractions!

The easiest way is to multiply the two denominators together. Don't forget to cross-multiply the numerators too!

COMMON MISTAKE: When subtracting a fraction, put brackets around its numerator — e.g. $9(2x - 1) - 4(x + 2) = 18x - 9 - 4x - 8$, distributing the minus sign to BOTH terms.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 **Non-Calculator**

Write $\frac{5}{x+3} + \frac{2}{x+4}$ as a single fraction.

STEP-BY-STEP SOLUTION:

Step 1: Make the denominators the same

Multiply the top and bottom of each fraction by the *other* fraction's denominator.

$$\frac{5(x+4)}{(x+3)(x+4)} + \frac{2(x+3)}{(x+4)(x+3)}$$

Step 2: Combine into a single fraction

Now they have the same denominator, we can add the tops!

$$\frac{5(x+4) + 2(x+3)}{(x+3)(x+4)}$$

Step 3: Expand and simplify the top

$$= \frac{5x + 20 + 2x + 6}{(x+3)(x+4)} = \frac{7x + 26}{(x+3)(x+4)}$$

EXAMPLE 2.2 **Non-Calculator**

Write $\frac{5}{x-3} - \frac{2}{x+4}$ as a single fraction.

STEP-BY-STEP SOLUTION:

Step 1: Make the denominators the same

$$\frac{5(x+4)}{(x-3)(x+4)} - \frac{2(x-3)}{(x+4)(x-3)}$$

Step 2: Combine into a single fraction

$$\frac{5(x+4) - 2(x-3)}{(x-3)(x+4)}$$

Step 3: Expand and simplify the top

Watch out for the negative sign when expanding $-2(x-3)$!

$$= \frac{5x + 20 - 2x + 6}{(x-3)(x+4)} = \frac{3x + 26}{(x-3)(x+4)}$$

3 Dividing Algebraic Fractions **CORE CONCEPT & METHOD**

To divide algebraic fractions, use the same rule as normal fractions: **Keep, Change, Flip (KFC)!**

Once it's a multiplication, factorise every part and cancel the matching brackets from the top and bottom.

COMMON MISTAKE: Flip **ONLY** the second fraction (the divisor) and change $\div$ to $\times$, then factorise every quadratic before multiplying.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 Non-Calculator

Simplify fully $\frac{4x + 12}{x - 5} \div \frac{2x^2 + 11x + 15}{x^2 - 5x}$

STEP-BY-STEP SOLUTION:

Step 1: Keep, Change, Flip

Keep the first fraction, change $\div$ to $\times$, and flip the second fraction upside down:

$$\frac{4x + 12}{x - 5} \times \frac{x^2 - 5x}{2x^2 + 11x + 15}$$

Step 2: Factorise EVERYTHING

Factorise all four parts (top left, bottom left, top right, bottom right):

$$\frac{4(x + 3)}{x - 5} \times \frac{x(x - 5)}{(2x + 5)(x + 3)}$$

Step 3: Cancel brackets

Since it's multiplication, any bracket on the top can cancel the same bracket on the bottom.

$$\frac{4(\textcolor{red}{x + 3})}{(\textcolor{blue}{x - 5})} \times \frac{x(\textcolor{blue}{x - 5})}{(2x + 5)(\textcolor{red}{x + 3})}$$

Result!

$$\frac{4x}{2x + 5}$$

EXAMPLE 3.2 **Non-Calculator**

Simplify fully $\frac{3x - 6}{x + 4} \div \frac{x^2 - 7x + 10}{2x^2 + 11x + 12}$

STEP-BY-STEP SOLUTION:**Step 1: Keep, Change, Flip**

$$\frac{3x - 6}{x + 4} \times \frac{2x^2 + 11x + 12}{x^2 - 7x + 10}$$

Step 2: Factorise EVERYTHING

$$\frac{3(x - 2)}{x + 4} \times \frac{(2x + 3)(x + 4)}{(x - 2)(x - 5)}$$

Step 3: Cancel bracketsThe $(x - 2)$ brackets cancel, and the $(x + 4)$ brackets cancel.

$$\frac{3(\cancel{x - 2})}{(\cancel{x + 4})} \times \frac{(2x + 3)(\cancel{x + 4})}{(\cancel{x - 2})(x - 5)}$$

Result!

$$\frac{3(2x + 3)}{x - 5}$$

4 Solving Algebraic Fractions **CORE CONCEPT & METHOD**When an algebraic fraction has an equals sign, your goal is to solve for x !**Step 1:** Combine the fractions into a single fraction (just like Step 2).**Step 2:** Multiply both sides by the denominator to get rid of it.**Step 3:** You will usually be left with a quadratic equation. Make it equal to 0, factorise, and solve!

COMMON MISTAKE: When multiplying through by the common denominator $(x - 2)(x + 3)$, remember to multiply the integer on the right-hand side by $(x - 2)(x + 3)$ as well.

MODEL WORKED EXAMPLES

EXAMPLE 4.1

 Non-Calculator

Solve $\frac{2}{2x-1} + \frac{8}{x+2} = 2$

STEP-BY-STEP SOLUTION:

Step 1: Combine over common denominator $(2x-1)(x+2)$

$$\frac{2(x+2) + 8(2x-1)}{(2x-1)(x+2)} = 2 \implies \frac{18x-4}{(2x-1)(x+2)} = 2$$

Step 2: Multiply both sides by $(2x-1)(x+2)$ and expand

$$18x-4 = 2(2x^2+3x-2) = 4x^2+6x-4$$

Step 3: Rearrange to = 0 and factorise

$$4x^2-12x=0 \implies 4x(x-3)=0 \implies x=0 \text{ or } x=3$$

EXAMPLE 4.2

 Non-Calculator

Solve $\frac{9}{x+2} - \frac{4}{2x-1} = 1$

STEP-BY-STEP SOLUTION:

Step 1: Combine over common denominator $(x+2)(2x-1)$

$$\frac{9(2x-1) - 4(x+2)}{(x+2)(2x-1)} = 1 \implies \frac{14x-17}{(x+2)(2x-1)} = 1$$

Step 2: Multiply both sides by $(x+2)(2x-1)$ and expand

$$14x-17 = 2x^2+3x-2$$

Step 3: Rearrange to = 0 and factorise

$$2x^2-11x+15=0 \implies (2x-5)(x-3)=0 \implies x=\frac{5}{2} \text{ or } x=3$$

Algebraic Proportion

1 Direct Proportion

□ CORE CONCEPT & METHOD

If $y \propto x$, it means $y = kx$ for some constant k . Substitute the given values to find k , then use the formula.

COMMON MISTAKE: Always write $y = kx$ and substitute the given pair to find k FIRST — do not guess additive patterns.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✎ Non-Calculator

y is directly proportional to x .

When $x = 6$, $y = 42$.

Find y when $x = 9$.

STEP-BY-STEP SOLUTION:

Direct proportion equation: $y = kx$

Substitute known values: $42 = k \times 6$

$$k = 42 \div 6 = 7$$

So, $y = 7x$

When $x = 9$, $y = 7 \times 9 = 63$.

2 Inverse Proportion

□ CORE CONCEPT & METHOD

If $y \propto \frac{1}{x}$, it means $y = \frac{k}{x}$. Find k by multiplying x and y together.

COMMON MISTAKE: Inverse proportion means $y = \frac{k}{x}$ so $k = x \times y$ (multiply the pair to find k), NEVER $k = \frac{y}{x}$.

MODEL WORKED EXAMPLES

EXAMPLE 2.1 **Non-Calculator**

y is inversely proportional to x .

When $x = 3$, $y = 12$.

Find y when $x = 9$.

STEP-BY-STEP SOLUTION:

Inverse proportion equation: $y = \frac{k}{x}$

Substitute known values: $12 = \frac{k}{3}$

$$k = 12 \times 3 = 36$$

$$\text{So, } y = \frac{36}{x}$$

$$\text{When } x = 9, y = \frac{36}{9} = 4.$$

3 Proportion with Powers & Roots **CORE CONCEPT & METHOD**

The steps are the same, just be careful to square, cube, or root the value of x first before finding k !

COMMON MISTAKE: Read the power or root carefully: "square of x " is x^2 , "square root of x " is $\sqrt{x}$, and apply the power ONLY to x ($y = kx^2$, NOT $(kx)^2$).

MODEL WORKED EXAMPLES

EXAMPLE 3.1 **Non-Calculator**

y is directly proportional to the square of x .

When $x = 4$, $y = 48$.

Find y when $x = 6$.

STEP-BY-STEP SOLUTION:

$$\text{Equation: } y = kx^2$$

$$\text{Substitute known values: } 48 = k(4^2)$$

$$48 = 16k$$

$$k = 48 \div 16 = 3$$

$$\text{So, } y = 3x^2$$

$$\text{When } x = 6, y = 3(6^2) = 3(36) = 108.$$

EXAMPLE 3.2 **Non-Calculator**

y is directly proportional to the cube root of x .

When $x = 27$, $y = 18$.

Find y when $x = 125$.

STEP-BY-STEP SOLUTION:

Equation: $y = k\sqrt[3]{x}$

Substitute known values: $18 = k(\sqrt[3]{27})$

$$18 = k(3)$$

$$k = 18 \div 3 = 6$$

$$\text{So, } y = 6\sqrt[3]{x}$$

$$\text{When } x = 125, y = 6(\sqrt[3]{125}) = 6(5) = 30.$$

4 Finding Relationships from Tables **CORE CONCEPT & METHOD**

To find the formula, test which calculation gives a constant k . Try $k = \frac{y}{x}$, then try $k = \frac{y}{x^2}$, etc.

COMMON MISTAKE: Test $\frac{y}{x}$, $\frac{y}{x^2}$, and $\frac{y}{x^3}$ across BOTH columns of the table — the relationship holds only when the constant k is identical for every pair.

MODEL WORKED EXAMPLES**EXAMPLE 4.1** **Non-Calculator**

A table shows values for x and y , where $y \propto x$. Find the missing value.

x	4	7
y	24	?

STEP-BY-STEP SOLUTION:

Since $y \propto x$, we know $y = kx$. Use the complete pair (4, 24) to find k :

$$24 = k(4) \implies k = 24 \div 4 = 6$$

The formula is $y = 6x$.

Now substitute $x = 7$ into the formula to find the missing value:

$$y = 6(7) = 42$$

Similar Shapes: Area, Volume & Frustums

1 Linear, Area & Volume Scale Factors

CORE CONCEPT & METHOD

If two 3D shapes are mathematically similar:

- **Linear Scale Factor (LSF)** = Ratio of Lengths
- **Area Scale Factor (ASF)** = LSF^2
- **Volume Scale Factor (VSF)** = LSF^3

Why?

Length is 1D (e.g. L)

Area is 2D (e.g. $L \times L = L^2$)

Volume is 3D (e.g. $L \times L \times L = L^3$)

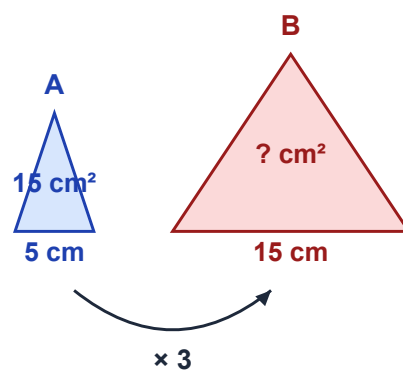
COMMON MISTAKE: If the length scale factor is k , multiply areas by k^2 and volumes by k^3 — never multiply area or volume directly by k .

MODEL WORKED EXAMPLES

EXAMPLE 1.1

 **Non-Calculator**

Triangle A has an area of 15 cm^2 and a base of 5 cm . Triangle B is mathematically similar with a base of 15 cm . What is the area of Triangle B?



STEP-BY-STEP SOLUTION:

Step 1: Find Linear Scale Factor (LSF)

$$\text{LSF} = \frac{\text{New Length}}{\text{Old Length}} = \frac{15}{5} = 3$$

Step 2: Find Area Scale Factor (ASF)

$$\text{ASF} = \text{LSF}^2 = 3^2 = 9$$

Step 3: Find Area of B

$$\text{Area}_B = 15 \times 9 = 135 \text{ cm}^2$$

2 Working Backwards

□ CORE CONCEPT & METHOD

If you are given the Areas, divide them to find the **ASF**. Then square root it to find the **LSF**!

If you are given Volumes, divide them to find the **VSF**, then cube root it to find the **LSF**.

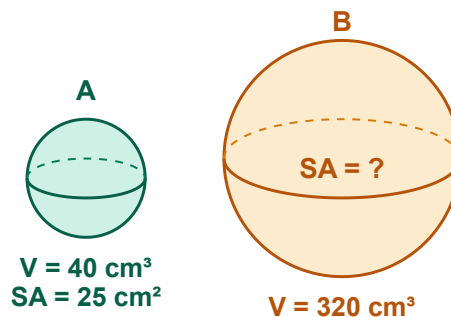
COMMON MISTAKE: When given two areas or two volumes, take the square root ($\sqrt{\text{ASF}}$) or cube root ($\sqrt[3]{\text{VSF}}$) FIRST to find the linear scale factor k before converting.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

✓ Non-Calculator

Two solid spheres are mathematically similar. The smaller sphere has a volume of 40 cm^3 and a surface area of 25 cm^2 . The larger sphere has a volume of 320 cm^3 . Find the surface area of the larger sphere.



STEP-BY-STEP SOLUTION:

Step 1: Find Volume Scale Factor (VSF)

$$\text{VSF} = \frac{320}{40} = 8$$

Step 2: Work Backwards to Linear Scale Factor (LSF)

$$\text{LSF} = \sqrt[3]{\text{VSF}} = \sqrt[3]{8} = 2$$

Step 3: Find Area Scale Factor (ASF)

$$\text{ASF} = \text{LSF}^2 = 2^2 = 4$$

Step 4: Find Surface Area of B

$$\text{SA}_B = 25 \times 4 = 100 \text{ cm}^2$$

3 Frustums

□ CORE CONCEPT & METHOD

A **frustum** is a cone with the top chopped off. To find its volume, calculate the volume of the **Large Cone** and subtract the volume of the **Small Cone** (the chopped part).

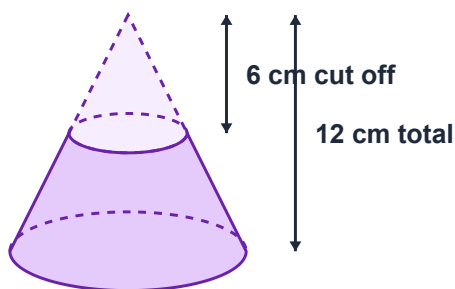
COMMON MISTAKE: A frustum is NOT a prism — always use similarity on the full large cone and removed small cone, then compute $V_{\text{frustum}} = V_{\text{large}} - V_{\text{small}}$.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

✓ Non-Calculator

A cone has a total height of 12 cm and a volume of 640 cm^3 . The top 6 cm of the cone is cut off to form a frustum. Find the volume of the frustum.



STEP-BY-STEP SOLUTION:

Step 1: Volumes of Similar Shapes

The small cone (top) is mathematically similar to the large cone (entire shape).

$$\text{LSF} = \frac{\text{Large Height}}{\text{Small Height}} = \frac{12}{6} = 2$$

$$\text{VSF} = 2^3 = 8$$

Step 2: Volume of Small Cone

Since the large cone is 8 times bigger, the small cone is $\frac{1}{8}$ of the large volume:

$$\text{Small Vol} = \frac{640}{8} = 80 \text{ cm}^3$$

Step 3: Volume of Frustum

The frustum is what remains when the small cone is removed from the large cone.

$$\text{Frustum Vol} = \text{Large Vol} - \text{Small Vol}$$

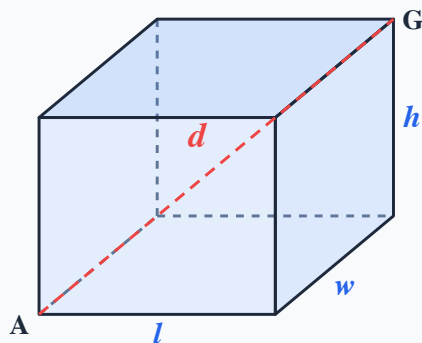
$$\text{Frustum Vol} = 640 - 80 = 560 \text{ cm}^3$$

3D Pythagoras & Trig

1 3D Pythagoras in Cuboids

□ CORE CONCEPT & METHOD

To find the space diagonal of a cuboid, you can use 3D Pythagoras: $d^2 = l^2 + w^2 + h^2$

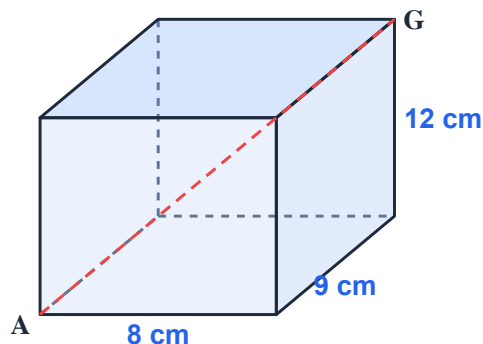


COMMON MISTAKE: Distinguish the 2D base diagonal $\sqrt{l^2 + w^2}$ from the 3D space diagonal $\sqrt{l^2 + w^2 + h^2}$, and keep the surd exact until the final step.

MODEL WORKED EXAMPLES

EXAMPLE 1.1 **Non-Calculator**

A cuboid has length 8 cm, width 9 cm, and height 12 cm. Find the length of the space diagonal AG.

**STEP-BY-STEP SOLUTION:**

Step 1: Identify the length, width, and height.

$$l = 8, w = 9, h = 12$$

Step 2: Use the 3D Pythagoras formula.

$$d^2 = l^2 + w^2 + h^2$$

$$AG^2 = 8^2 + 9^2 + 12^2$$

$$AG^2 = 64 + 81 + 144$$

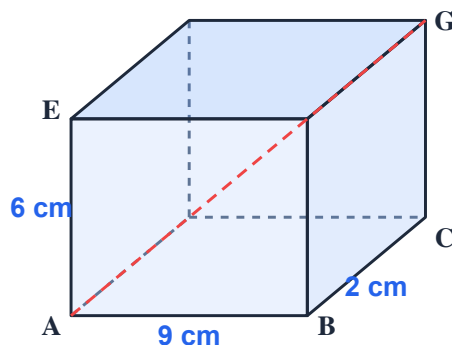
$$AG^2 = 289$$

Step 3: Square root to find the space diagonal.

$$AG = \sqrt{289} = 17 \text{ cm}$$

**EXAMPLE 1.2**

A cuboid ABCDEFGH has $AB = 9$ cm, $BC = 2$ cm, $AE = 6$ cm. Calculate the length of the space diagonal AG, correct to 1 decimal place.

**STEP-BY-STEP SOLUTION:**

Step 1: Identify dimensions from the letters.

Length (AB) = 9, Width (BC) = 2, Height (AE) = 6

Step 2: Use the 3D Pythagoras formula.

$$d^2 = l^2 + w^2 + h^2$$

$$AG^2 = 9^2 + 2^2 + 6^2$$

$$AG^2 = 81 + 4 + 36$$

$$AG^2 = 121$$

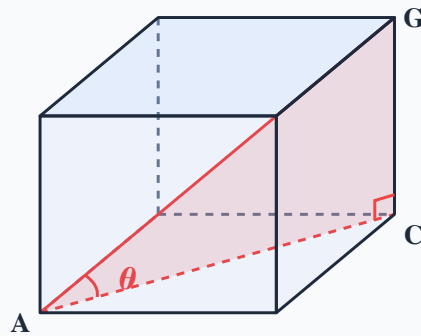
Step 3: Square root to find the space diagonal.

$$AG = \sqrt{121} = 11 \text{ cm}$$

2 3D Trigonometry

▣ CORE CONCEPT & METHOD

First, find the 2D diagonal using Pythagoras. Then, use SOH CAH TOA with the vertical height to find the angle of elevation.



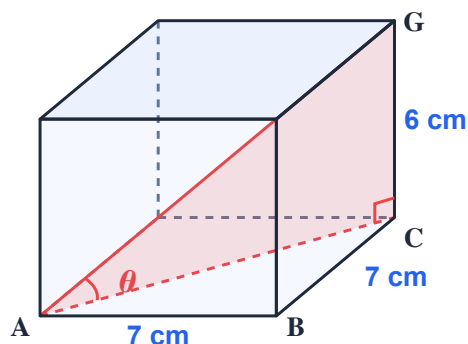
COMMON MISTAKE: Draw out the flat 2D right-angled triangle inside the solid and label its right angle clearly before choosing $\sin$, $\cos$, or $\tan$.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 Calculator

A cuboid $ABCDEFGH$ has $AB = 7$ cm, $BC = 7$ cm, and $AE = 6$ cm. Find the angle the space diagonal AG makes with the base $ABCD$.



STEP-BY-STEP SOLUTION:

Step 1: Find the 2D diagonal of the base (AC).

Look at the base $ABCD$. It's a 7×7 rectangle.

Use Pythagoras:

$$AC^2 = 7^2 + 7^2 = 98$$

$$AC = \sqrt{98} \text{ cm}$$

Step 2: Draw out the vertical right-angled triangle ACG.

We want the angle of elevation θ at A .

- Adjacent (base AC) = $\sqrt{98}$

- Opposite (height CG) = 6

Step 3: Use SOH CAH TOA.

$$\tan(\theta) = \frac{\text{Opp}}{\text{Adj}}$$

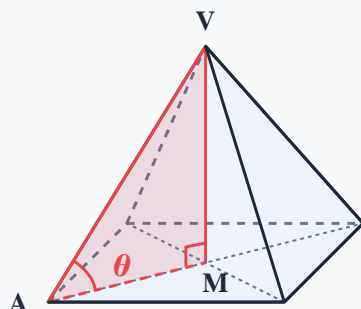
$$\tan(\theta) = \frac{6}{\sqrt{98}}$$

$$\theta = \tan^{-1}\left(\frac{6}{\sqrt{98}}\right) = 31.2^\circ \text{ (1 d.p.)}$$

3 Prisms and Pyramids

□ CORE CONCEPT & METHOD

For pyramids, find the diagonal of the base first, then halve it to find the distance to the center. Then use right-angled triangles with the vertical height.



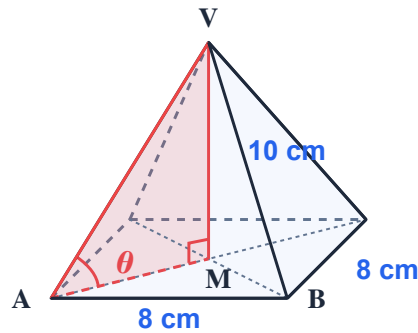
COMMON MISTAKE: In a right pyramid, the vertical height drops to the CENTRE M of the base, so use HALF the base diagonal ($AM = \frac{1}{2}AC$) with the sloping edge VA .

MODEL WORKED EXAMPLES

EXAMPLE 3.1



A pyramid has a square base ABCD of side length 8 cm. The vertical height of the pyramid from the center of the base (M) to the apex (V) is 10 cm. Find the angle the edge AV makes with the base ABCD.



STEP-BY-STEP SOLUTION:

Step 1: Find the full diagonal of the square base (AC).

The base is an 8×8 square.

$$AC^2 = 8^2 + 8^2 = 64 + 64 = 128$$

$$AC = \sqrt{128} \text{ cm}$$

Step 2: Find the distance from the corner to the center (AM).

The center M is exactly halfway along the diagonal AC .

$$AM = \frac{\sqrt{128}}{2} \text{ cm}$$

Step 3: Use SOH CAH TOA in the vertical triangle AMV.

We want the angle θ at A .

$$\text{- Adjacent (AM)} = \frac{\sqrt{128}}{2}$$

$$\text{- Opposite (height VM)} = 10$$

$$\tan(\theta) = \frac{\text{Opp}}{\text{Adj}} = \frac{10}{\frac{\sqrt{128}}{2}}$$

$$\theta = \tan^{-1}\left(\frac{10}{5.6568\dots}\right) = 60.5^\circ \text{ (1 d.p.)}$$

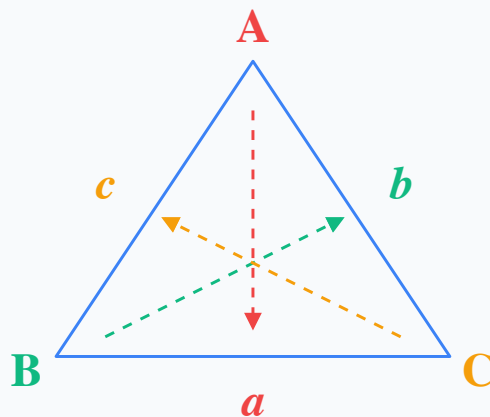
The Sine Rule

1 Labelling the Triangle

□ CORE CONCEPT & METHOD

Before using the Sine, Cosine, or Area Rules, you must know how to label the triangle correctly.

- **Angles** are labelled with capital letters: A , B , C .
- **Sides** are labelled with lower-case letters: a , b , c .
- **Crucial Rule:** Side a is ALWAYS exactly opposite angle A . Side b is opposite angle B , and side c is opposite angle C .



COMMON MISTAKE: Lowercase side a is ALWAYS directly opposite uppercase angle A — mislabelling adjacent sides breaks both the Sine and Cosine Rules.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

If you are given angle B , which side is exactly opposite to it?

✎ Non-Calculator

STEP-BY-STEP SOLUTION:

Side b is always opposite angle B .

2 Finding Missing Sides

□ CORE CONCEPT & METHOD

The Sine Rule states: $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$.

Use this to find a missing side when you know a side and its opposite angle, and another angle.

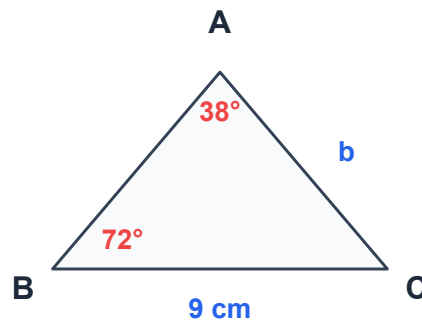
COMMON MISTAKE: Use the Sine Rule only when you have a complete matching opposite pair (an angle and its opposite side); put the unknown side on TOP ($\frac{a}{\sin A} = \frac{b}{\sin B}$).

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 **Calculator**

In triangle ABC, angle A = 38°, angle B = 72°, and side a = 9 cm. Find side b.



STEP-BY-STEP SOLUTION:

We know side a opposite angle A , and we want side b opposite angle B .

Write the Sine Rule:

$$\frac{b}{\sin(B)} = \frac{a}{\sin(A)}$$

Substitute the values:

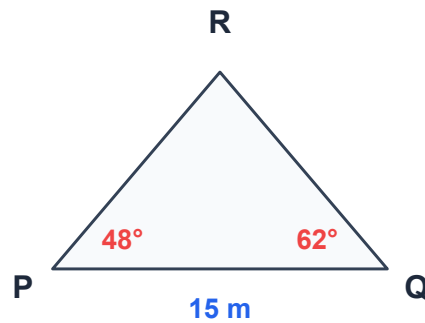
$$\frac{b}{\sin(72^\circ)} = \frac{9}{\sin(38^\circ)}$$

Multiply both sides by $\sin(72^\circ)$:

$$b = \frac{9 \times \sin(72^\circ)}{\sin(38^\circ)} = \frac{9 \times 0.9511...}{0.6157...} = 13.9 \text{ cm (1 d.p.)}$$

 **Calculator**
EXAMPLE 2.2

Triangle PQR has angle P = 48° , angle Q = 62° , and PQ = 15 m. Find side QR.

**STEP-BY-STEP SOLUTION:**

Side QR is opposite angle P (= 48°). Side PQ is opposite angle R.

First find angle R:

$$R = 180^\circ - 48^\circ - 62^\circ = 70^\circ$$

Now apply the Sine Rule:

$$\frac{QR}{\sin(48^\circ)} = \frac{15}{\sin(70^\circ)}$$

$$QR = \frac{15 \times \sin(48^\circ)}{\sin(70^\circ)} = \frac{15 \times 0.7431\dots}{0.9397\dots} = 11.9 \text{ m (1 d.p.)}$$

3 Finding Missing Angles

CORE CONCEPT & METHOD

Flip the rule to find angles: $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$.

Remember, if the question says the angle is **obtuse**, subtract your calculator answer from 180° !

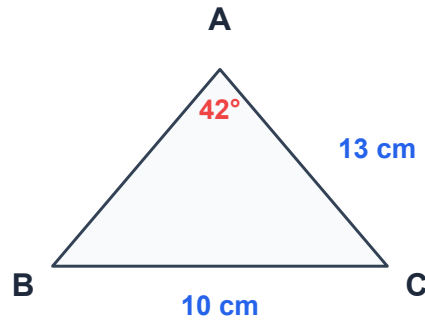
COMMON MISTAKE: Put $\sin x$ on TOP when finding an angle, and if the question states the angle is OBTUSE, subtract your acute $\sin^{-1}$ answer from 180° .

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 **Calculator**

In triangle ABC, side $a = 10$ cm, side $b = 13$ cm, and angle $A = 42^\circ$. Find angle B.



STEP-BY-STEP SOLUTION:

Use the flipped Sine Rule:

$$\frac{\sin(B)}{b} = \frac{\sin(A)}{a}$$

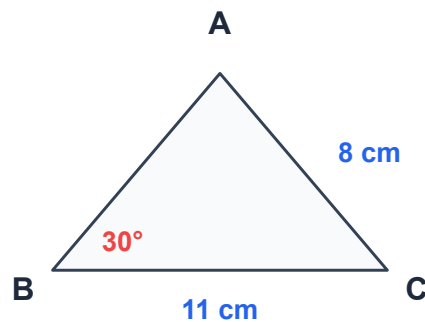
Substitute:

$$\begin{aligned}\frac{\sin(B)}{13} &= \frac{\sin(42^\circ)}{10} \\ \sin(B) &= \frac{13 \times \sin(42^\circ)}{10} = \frac{13 \times 0.6691\dots}{10} = 0.8699\dots \\ B &= \sin^{-1}(0.8699\dots) = 60.4^\circ \text{ (1 d.p.)}\end{aligned}$$

EXAMPLE 3.2

 **Calculator**

In triangle ABC, side $a = 11$ cm, side $b = 8$ cm, and angle $B = 30^\circ$. Angle A is obtuse. Find angle A.



STEP-BY-STEP SOLUTION:

Use the Sine Rule:

$$\begin{aligned}\frac{\sin(A)}{a} &= \frac{\sin(B)}{b} \\ \sin(A) &= \frac{11 \times \sin(30^\circ)}{8} = \frac{11 \times 0.5}{8} = 0.6875\end{aligned}$$

The calculator gives $\sin^{-1}(0.6875) = 43.4^\circ$, but angle A is **obtuse**, so:

$$A = 180^\circ - 43.4^\circ = 136.6^\circ \text{ (1 d.p.)}$$

4 Problem Solving

□ CORE CONCEPT & METHOD

Combine the Sine Rule with the Area of a Triangle formula ($\frac{1}{2}ab \sin(C)$), Pythagoras, or angles in a triangle adding to 180° .

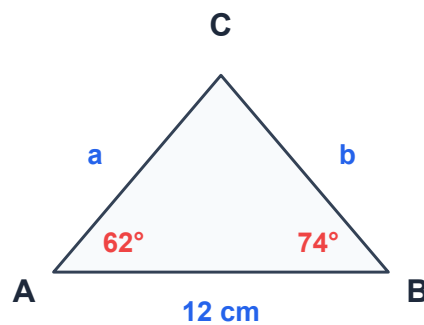
COMMON MISTAKE: Remember angles in a triangle sum to 180° — you often need to subtract the two given angles from 180° first to unlock a matching opposite pair.

MODEL WORKED EXAMPLES

EXAMPLE 4.1

 **Calculator**

Triangle ABC has angle A = 62° , angle B = 74° , and side AB = 12 cm. Find the perimeter of the triangle.



STEP-BY-STEP SOLUTION:

Step 1 — Find angle C:

$$C = 180^\circ - 62^\circ - 74^\circ = 44^\circ$$

Step 2 — Side AB (= c) is opposite angle C = 44° . Use Sine Rule to find BC (= a , opposite A = 62°):

$$\frac{a}{\sin(62^\circ)} = \frac{12}{\sin(44^\circ)} \implies a = \frac{12 \sin(62^\circ)}{\sin(44^\circ)} = 15.25... \text{ cm}$$

Step 3 — Find AC (= b , opposite B = 74°):

$$b = \frac{12 \sin(74^\circ)}{\sin(44^\circ)} = 16.61... \text{ cm}$$

Step 4 — Perimeter:

$$12 + 15.25... + 16.61... = 43.9 \text{ cm (1 d.p.)}$$

The Cosine Rule

1 Finding Missing Sides

□ CORE CONCEPT & METHOD

The Cosine Rule states: $a^2 = b^2 + c^2 - 2bc \cos(A)$.

Use this to find a missing side when you know two sides and the *included angle* (the angle between them).

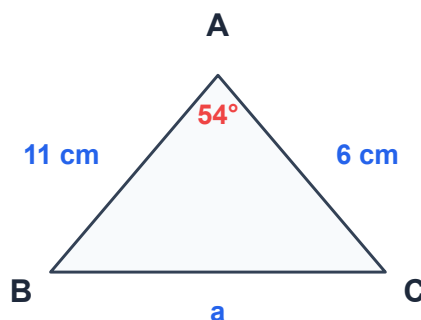
COMMON MISTAKE: In $a^2 = b^2 + c^2 - 2bc \cos A$, remember to take the SQUARE ROOT at the end, and note that when $A > 90^\circ$, $\cos A$ is negative so $-2bc \cos A$ ADDS to $b^2 + c^2$.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

 **Calculator**

A triangle has sides $b = 6$ cm and $c = 11$ cm with an included angle $A = 54^\circ$. Find side a .



STEP-BY-STEP SOLUTION:

Apply the Cosine Rule:

$$a^2 = b^2 + c^2 - 2bc \cos(A)$$

Substitute the values:

$$a^2 = 6^2 + 11^2 - 2(6)(11) \cos(54^\circ)$$

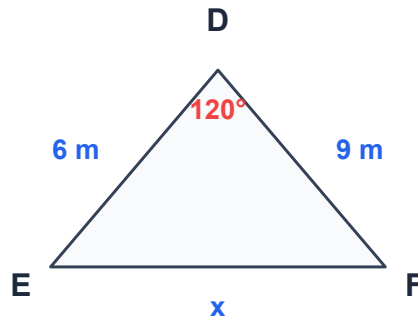
$$a^2 = 36 + 121 - 132 \times 0.5878 \dots$$

$$a^2 = 157 - 77.59 \dots = 79.41 \dots$$

$$a = \sqrt{79.41 \dots} = 8.9 \text{ cm (1 d.p.)}$$

**EXAMPLE 1.2**

In triangle DEF, DE = 6 m, DF = 9 m, and angle D = 120°. Find EF.

**STEP-BY-STEP SOLUTION:**

Here $b = 9$, $c = 6$, and the included angle is $D = 120^\circ$:

$$d^2 = 9^2 + 6^2 - 2(9)(6) \cos(120^\circ)$$

Since $\cos(120^\circ) = -0.5$:

$$d^2 = 81 + 36 - 108 \times (-0.5) = 117 + 54 = 171$$

$$d = \sqrt{171} = 13.1 \text{ m (1 d.p.)}$$

2 Finding Missing Angles

□ CORE CONCEPT & METHOD

Rearrange the rule to find angles: $\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$.

Use this when you know all three sides of the triangle.

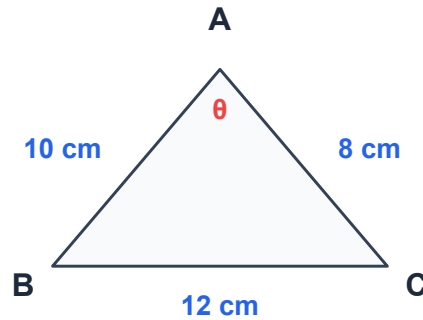
COMMON MISTAKE: In $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, the subtracted side a^2 MUST be the side opposite the angle A you are trying to find.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 Calculator

A triangle has sides $a = 12$ cm, $b = 8$ cm, and $c = 10$ cm. Find angle A.



STEP-BY-STEP SOLUTION:

Use the rearranged Cosine Rule:

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

Substitute:

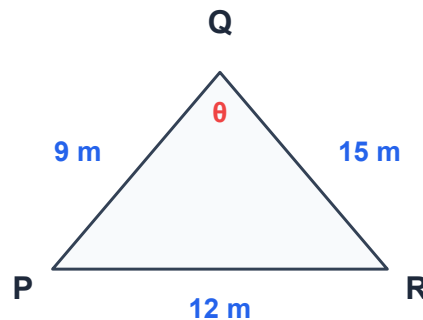
$$\cos(A) = \frac{8^2 + 10^2 - 12^2}{2 \times 8 \times 10} = \frac{64 + 100 - 144}{160} = \frac{20}{160} = 0.125$$

$$A = \cos^{-1}(0.125) = 82.8^\circ \text{ (1 d.p.)}$$

EXAMPLE 2.2

 Calculator

Triangle PQR has $PQ = 9$ m, $QR = 15$ m, $PR = 12$ m. Find angle Q.



STEP-BY-STEP SOLUTION:

Angle Q is between sides PQ and QR, so the side opposite is $PR = 12$ m. Let $a = PR = 12$, $b = PQ = 9$, $c = QR = 15$:

$$\cos(Q) = \frac{9^2 + 15^2 - 12^2}{2 \times 9 \times 15} = \frac{81 + 225 - 144}{270} = \frac{162}{270} = 0.6$$

$$Q = \cos^{-1}(0.6) = 53.1^\circ \text{ (1 d.p.)}$$

3 Problem Solving

□ CORE CONCEPT & METHOD

You may need to form quadratic equations if the sides are given algebraically, or combine the Cosine Rule with the Area formula ($\frac{1}{2}ab \sin(C)$).

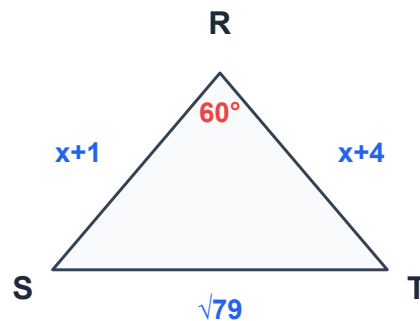
COMMON MISTAKE: When sides are algebraic expressions like $(2x - 4)$ and $(x + 1)$ with 60° , substitute $\cos 60^\circ = \frac{1}{2}$ so $-2bc \cos 60^\circ$ simplifies cleanly to $-bc$.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 **Non-Calculator**

A triangle has sides $(x + 1)$ cm, $(x + 4)$ cm and $\sqrt{79}$ cm. The angle between the first two sides is 60° . Find x .



STEP-BY-STEP SOLUTION:

The side opposite the 60° angle is $\sqrt{79}$. Apply the Cosine Rule:

$$(\sqrt{79})^2 = (x + 1)^2 + (x + 4)^2 - 2(x + 1)(x + 4) \cos(60^\circ)$$

Since $\cos(60^\circ) = 0.5$:

$$79 = (x^2 + 2x + 1) + (x^2 + 8x + 16) - (x + 1)(x + 4)$$

$$79 = 2x^2 + 10x + 17 - (x^2 + 5x + 4)$$

$$79 = x^2 + 5x + 13$$

$$x^2 + 5x - 66 = 0$$

$$(x + 11)(x - 6) = 0$$

Since lengths must be positive, $x = 6$.

Area of Any Triangle

1 Finding the Area

□ CORE CONCEPT & METHOD

To find the area of *any* triangle, use the formula: $\text{Area} = \frac{1}{2}ab \sin(C)$.

You need two sides (a and b) and the *included angle* (C) between them.

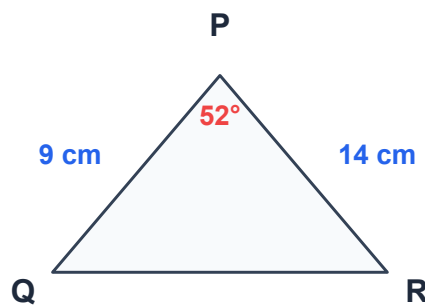
COMMON MISTAKE: In $\text{Area} = \frac{1}{2}ab \sin C$, angle C MUST be the included angle sandwiched BETWEEN sides a and b — never use $\frac{1}{2} \times \text{base} \times \text{slanted side}$.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

[Calculator](#)

A triangle has two sides of length 9 cm and 14 cm with an included angle of 52° . Find its area.



STEP-BY-STEP SOLUTION:

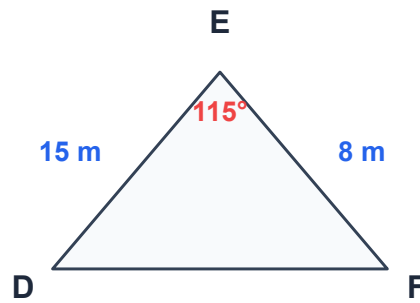
Use the formula $\text{Area} = \frac{1}{2}ab \sin(C)$.

Substitute $a = 9$, $b = 14$, $C = 52^\circ$:

$$\begin{aligned} \text{Area} &= \frac{1}{2} \times 9 \times 14 \times \sin(52^\circ) \\ &= \frac{1}{2} \times 9 \times 14 \times 0.7880\dots \\ &= 49.6 \text{ cm}^2 \text{ (1 d.p.)} \end{aligned}$$

**EXAMPLE 1.2**

Triangle DEF has $DE = 15\text{ m}$, $EF = 8\text{ m}$, and angle $E = 115^\circ$. Find the area.

**STEP-BY-STEP SOLUTION:**

$a = DE = 15$, $b = EF = 8$, included angle $E = 115^\circ$:

$$\begin{aligned}\text{Area} &= \frac{1}{2} \times 15 \times 8 \times \sin(115^\circ) \\ &= \frac{1}{2} \times 120 \times 0.9063\dots \\ &= 54.4\text{ m}^2 \text{ (1 d.p.)}\end{aligned}$$

2 Finding a Missing Angle

□ CORE CONCEPT & METHOD

If you know the area and two sides, you can find the included angle:

$$\sin(C) = \frac{\text{Area}}{\frac{1}{2}ab}$$

Don't forget to use inverse sine ($\sin^{-1}$) at the end!

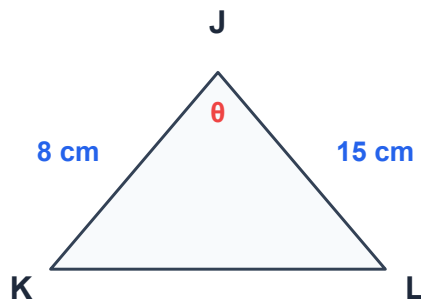
COMMON MISTAKE: After rearranging to $\sin C = \frac{2 \times \text{Area}}{ab}$, do not forget to apply $\sin^{-1}$ to obtain the angle in degrees.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 **Calculator**

A triangle has sides of 8 cm and 15 cm. Its area is 42 cm^2 . Find the included angle between the two sides.



STEP-BY-STEP SOLUTION:

Start with the formula:

$$\text{Area} = \frac{1}{2}ab \sin(C)$$

Substitute the known values:

$$\begin{aligned} 42 &= \frac{1}{2} \times 8 \times 15 \times \sin(C) \\ 42 &= 60 \sin(C) \\ \sin(C) &= \frac{42}{60} = 0.7 \\ C &= \sin^{-1}(0.7) = 44.4^\circ \text{ (1 d.p.)} \end{aligned}$$

EXAMPLE 2.2

 **Calculator**

Two sides of a triangle measure 12 m and 10 m. The area of the triangle is 45 m^2 . Find the angle between these sides.

STEP-BY-STEP SOLUTION:

$$\begin{aligned} 45 &= \frac{1}{2} \times 12 \times 10 \times \sin(C) \\ 45 &= 60 \sin(C) \\ \sin(C) &= \frac{45}{60} = 0.75 \\ C &= \sin^{-1}(0.75) = 48.6^\circ \text{ (1 d.p.)} \end{aligned}$$

3 Problem Solving

CORE CONCEPT & METHOD

Sometimes the sides are given as algebraic expressions or ratios. You'll need to set up a quadratic or linear equation and solve for x !

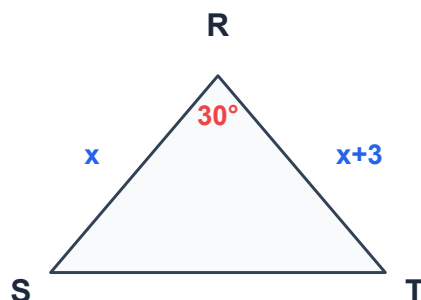
COMMON MISTAKE: Use exact trig values ($\sin 30^\circ = \frac{1}{2}$, $\sin 60^\circ = \frac{\sqrt{3}}{2}$) when forming a quadratic from the area, and reject any negative root since side lengths must be positive.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 Non-Calculator

The area of a triangle is 10 cm^2 . Two sides are $(x) \text{ cm}$ and $(x + 3) \text{ cm}$, and the included angle is 30° . Find x .



STEP-BY-STEP SOLUTION:

Use the Area formula:

$$10 = \frac{1}{2} \times x \times (x + 3) \times \sin(30^\circ)$$

Substitute $\sin(30^\circ) = 0.5$:

$$10 = \frac{1}{2} \times x(x + 3) \times 0.5 = \frac{1}{4}x(x + 3)$$

Multiply both sides by 4:

$$\begin{aligned} x(x + 3) &= 40 \\ x^2 + 3x - 40 &= 0 \\ (x + 8)(x - 5) &= 0 \end{aligned}$$

Since lengths must be positive, $x = 5 \text{ cm}$.

Conditional Probability

1 Without Replacement (Items)

□ CORE CONCEPT & METHOD

When an item is taken and **not replaced**, the probabilities change for the second pick!

If you take a Red counter from a bag of 10, there are now only 9 counters left in total, and 1 fewer Red counter.

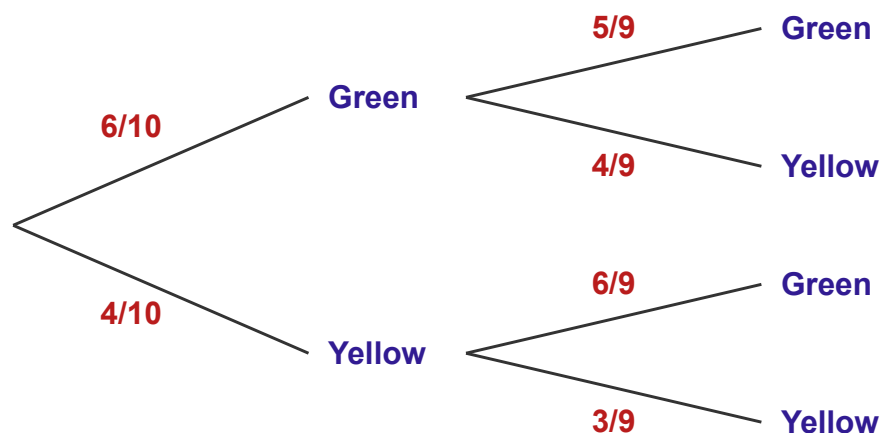
COMMON MISTAKE: Without replacement, reduce the total denominator by 1 on EVERY second-pick branch, and reduce the numerator by 1 ONLY for the colour already picked first.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✓ Non-Calculator

A box contains 6 green marbles and 4 yellow marbles. Two marbles are taken out **without replacement**. Find the probability that both marbles are green.



STEP-BY-STEP SOLUTION:

First pick: There are 10 marbles in total, 6 of which are green.

$$P(\text{Green}_1) = \frac{6}{10}$$

Second pick: One green marble has been removed, so there are now 5 green marbles left out of 9 total.

$$P(\text{Green}_2 \mid \text{Green}_1) = \frac{5}{9}$$

Both green: Multiply along the branches:

$$P(\text{both green}) = \frac{6}{10} \times \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$$

2 Given Probabilities

□ CORE CONCEPT & METHOD

Sometimes you are explicitly told how the probability changes. For example: 'If it rains today, the probability it rains tomorrow is 0.8. If it doesn't, the probability is 0.3'.

Use these specific numbers on the branches of your tree diagram!

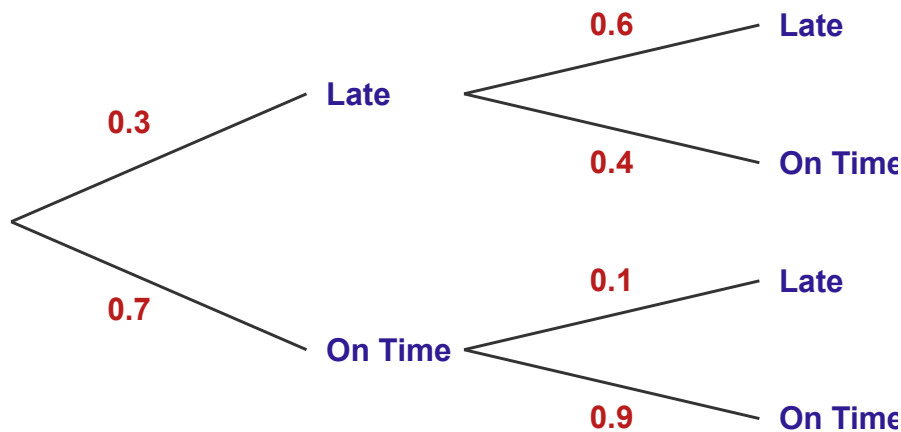
COMMON MISTAKE: Each pair of branches from a single node must add up to 1 — check the top second-stage pair and bottom second-stage pair separately.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

✎ Non-Calculator

The probability that a bus is late on Wednesday is 0.3. If it is late on Wednesday, the probability it is late on Thursday is 0.6. If it is on time on Wednesday, the probability it is late on Thursday is 0.1. Find the probability that the bus is late on **both** days.



STEP-BY-STEP SOLUTION:

Draw a tree diagram with two branches on Wednesday (Late / On Time), then two branches on Thursday for each case.

P(Late Wednesday AND Late Thursday) — multiply along the *Late, Late* branch:

$$0.3 \times 0.6 = \mathbf{0.18}$$

3 Without Replacement (Numbers)

□ CORE CONCEPT & METHOD

If you take a number card and don't replace it, the pool of available numbers shrinks. You must count how many successful outcomes are left for the second pick.

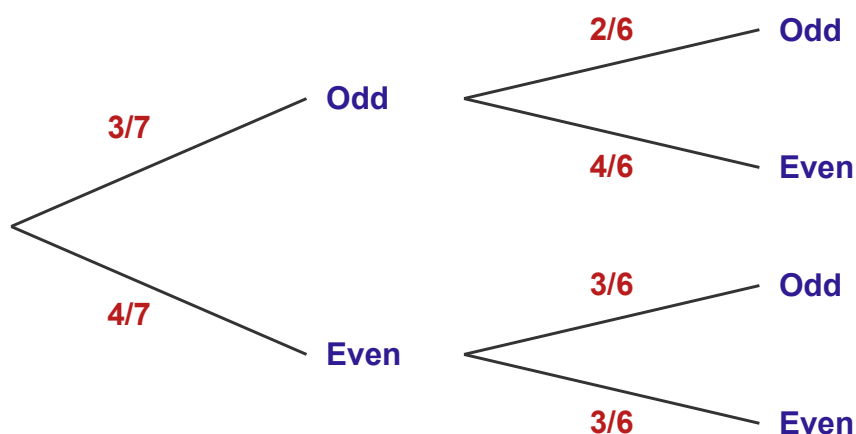
COMMON MISTAKE: Multiply along branches for "AND", then ADD the separate branch products for "different colours" or use $1 - P(\text{none})$ for "at least one".

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 Non-Calculator

A set of tiles has the numbers 2, 3, 4, 5, 6, 7, 8. Two tiles are drawn at random **without replacement**. Find the probability that both numbers drawn are ODD.



STEP-BY-STEP SOLUTION:

First draw: There are 7 tiles in total. The odd numbers are 3, 5, and 7 (so there are 3 odd tiles).

$$P(\text{Odd}_1) = \frac{3}{7}$$

Second draw: One odd tile is gone. There are 2 odd tiles left out of 6 remaining tiles.

$$P(\text{Odd}_2 \mid \text{Odd}_1) = \frac{2}{6}$$

Both odd: Multiply along the branches:

$$P(\text{both odd}) = \frac{3}{7} \times \frac{2}{6} = \frac{6}{42} = \frac{1}{7}$$

Histograms

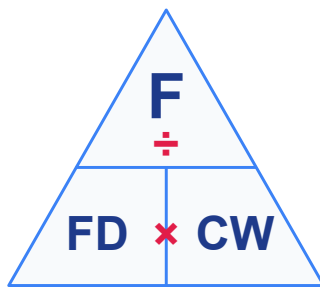
1 Frequency Density

□ CORE CONCEPT & METHOD

Unlike bar charts, histograms are used for continuous data with **unequal class widths**.

To make it fair, we plot **Frequency Density (FD)** on the y-axis, not frequency!

Remember the FD triangle! Cover up the one you want to find.



$$\text{Frequency} = \mathbf{FD} \times \mathbf{Class\ Width}$$

$$\mathbf{FD} = \text{Frequency} \div \mathbf{Class\ Width}$$

$$\mathbf{Class\ Width} = \text{Frequency} \div \mathbf{FD}$$

COMMON MISTAKE: Always plot Frequency Density = $\frac{\text{Frequency}}{\text{Class Width}}$ on the vertical axis — NEVER plot raw frequency when class widths are unequal.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

A grouped frequency table has the interval $30 < x \leq 50$ with a frequency of **84**. Calculate the frequency density for this bar.

STEP-BY-STEP SOLUTION:

Step 1 — Find the class width:

$$\text{Class width} = 50 - 30 = 20$$

Step 2 — Apply the formula:

$$\text{Frequency Density} = \frac{\text{Frequency}}{\text{Class Width}} = \frac{84}{20} = 4.2$$

Answer: The bar for this interval has height **4.2** on the frequency density axis.

2 Frequencies from Areas

▣ CORE CONCEPT & METHOD

On a histogram, the **AREA** of the bar represents the Frequency.

$$\text{Frequency} = \text{Frequency Density} \times \text{Class Width}$$

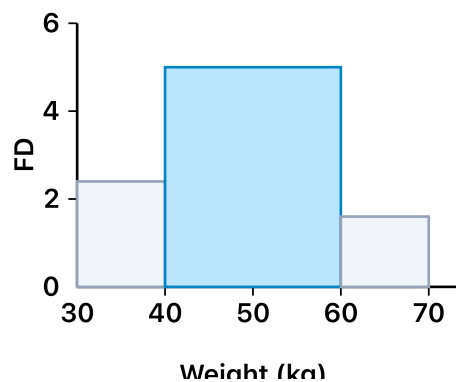
COMMON MISTAKE: In a histogram, Frequency = Area of bar = Class Width $\times$ Frequency Density — if the y -axis is unnumbered, use a bar with known frequency to find the scale first.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 Non-Calculator

A histogram shows the distribution of weights.



Find the frequency for the interval $40 < w \leq 60$.

STEP-BY-STEP SOLUTION:

Step 1 — Identify Class Width (Width of the bar):

The bar starts at 40 and ends at 60.

$$\text{Class Width} = 60 - 40 = 20.$$

Step 2 — Identify Frequency Density (Height of the bar):

The bar goes up to 5 on the y -axis.

$$\text{FD} = 5$$

Step 3 — Calculate Frequency (Area of the bar):

$$\text{Frequency} = \text{FD} \times \text{Class Width}$$

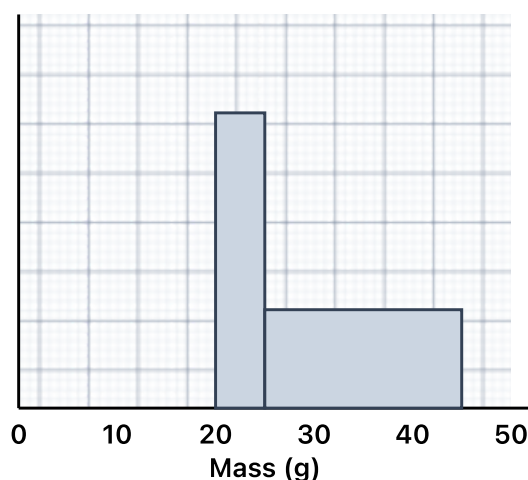
$$\text{Frequency} = 5 \times 20 = 100$$

Answer: There are 100 items in this interval.

EXAMPLE 2.2 **Non-Calculator**

A partially completed frequency table and histogram show the mass of some pears. The y-axis has no scale.

Mass, m (g)	Frequency
$0 < m \leq 10$	24
$10 < m \leq 20$	60
$20 < m \leq 25$	?
$25 < m \leq 45$	40
$45 < m \leq 50$	16



Calculate the missing frequency for the interval $20 < m \leq 25$.

STEP-BY-STEP SOLUTION:

Step 1 — Deduce the scale from the known bar ($25 < m \leq 45$):

Frequency = 40

Class Width = $45 - 25 = 20$

$$\text{FD} = \text{Frequency} \div \text{Class Width} = 40 \div 20 = 2$$

On the graph, this bar is exactly **2 large squares** tall. So, 2 large squares = an FD of 2, meaning **1 large square = an FD of 1**.

Step 2 — Find the FD for the missing bar ($20 < m \leq 25$):

On the graph, this bar is **6 large squares** tall.

$$\text{FD} = 6 \times 1 = 6$$

Step 3 — Calculate the missing frequency:

Class Width = $25 - 20 = 5$

$$\text{Frequency} = \text{FD} \times \text{Class Width} = 6 \times 5 = 30$$

Answer: The missing frequency is 30.

3 Estimating Proportions

CORE CONCEPT & METHOD

If a bar goes from 10 to 20 with a frequency of 40, we assume those 40 items are spread evenly.

To estimate how many are between 15 and 20 (which is half the bar's width), you take half the frequency (20).

COMMON MISTAKE: When splitting a bar at a value (or finding the median), take that fraction of the class width multiplied by its Frequency Density, assuming values are evenly distributed across the class.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

 **Non-Calculator**

A histogram bar for the interval $10 < t \leq 50$ has a frequency of **180**.

Estimate the frequency of items between **20 and 30**.

STEP-BY-STEP SOLUTION:

Step 1 — Find the total width of the bar:

Class Width = $50 - 10 = 40$.

Step 2 — Find the width of the target section:

We want the section from 20 to 30.

Target Width = $30 - 20 = 10$.

Step 3 — Find the proportion:

The target section is $\frac{10}{40}$ (or one-quarter) of the entire bar.

Step 4 — Calculate the estimated frequency:

Assuming the 180 items are spread evenly, we take one-quarter of them.

$$180 \div 4 = 45$$

Answer: The estimated frequency is 45.