

GCSE MATHEMATICS (9-1) — TUTORIALS & WORKED EXAMPLES

Grade 8 & 9 Mathematics — Tutorials & Examples

(Higher Tier)

Higher Tier Tutorials, Common Mistake Warnings & Step-by-Step Worked Examples

- Higher Tier Core Concept Notes & Precision Geometric/Graph Diagrams
- Exam-Focused "Common Mistake" Warnings on every single step
- Model Worked Examples tagged **Calculator** or **Non-Calculator**
- Streamlined Higher Revision Guide (Tutorials & Worked Examples Only)

Student Name: _____

Class / Teacher: _____

Target Grade: _____



SCAN FOR ONLINE PRACTICE & WORKED SOLUTIONS

BreakingDownMaths.com

Interactive Step-by-Step GCSE Mathematics Mastery

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ALGEBRA — QUADRATICS

Completing the Square

1 Completing the Square

□ CORE CONCEPT & METHOD

Completing the square allows us to rewrite a quadratic to easily find its turning point.

Halve the middle number (b) and put it in a bracket squared: $(x + \frac{b}{2})^2$

Then **subtract** the square of that number, and add the original c back on the end!

COMMON MISTAKE: When $a \neq 1$ in $ax^2 + bx + c$, factor a out of the first two terms **FIRST** and remember to multiply the subtracted square by a when expanding the outer bracket.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✎ Non-Calculator

Write $x^2 + 8x + 3$ in the form $(x + a)^2 + b$

STEP-BY-STEP SOLUTION:

1. Halve the middle number ($8 \div 2 = 4$). Put in a bracket: $(x + 4)^2$

2. Subtract 4^2 : $(x + 4)^2 - 16$

3. Bring down the original $+3$:

$$(x + 4)^2 - 16 + 3$$

$$(x + 4)^2 - 13$$

EXAMPLE 1.2

✎ Non-Calculator

Write $2x^2 + 16x + 9$ in the form $a(x + b)^2 + c$

STEP-BY-STEP SOLUTION:

1. Factor out the 2 from the x terms first:

$$2(x^2 + 8x) + 9$$

2. Complete the square inside the bracket (halve the 8 to get 4):

$$2[(x + 4)^2 - 4^2] + 9$$

$$2[(x + 4)^2 - 16] + 9$$

3. Expand the outer bracket by multiplying the 2 back in:

$$2(x + 4)^2 - 32 + 9$$

4. Simplify the numbers on the end:

$$2(x + 4)^2 - 23$$

EXAMPLE 1.3 **Non-Calculator**

Write $x^2 - 10x + 7$ in the form $(x + a)^2 + b$

STEP-BY-STEP SOLUTION:

1. Halve the middle number ($-10 \div 2 = -5$). Put in a bracket: $(x - 5)^2$

2. Subtract $(-5)^2$. (Remember, negative squared is always positive, so we still subtract 25):
 $(x - 5)^2 - 25$

3. Bring down the original +7:
 $(x - 5)^2 - 25 + 7$

$$(x - 5)^2 - 18$$

2 Solving by Completing the Square **CORE CONCEPT & METHOD**

To solve a quadratic equation by completing the square, first rewrite the quadratic in completed square form $(x + a)^2 + b = 0$, then make x the subject!

1. Complete the square on the quadratic expression.
2. Move the loose constant to the other side.
3. Square root both sides (don't forget the $\pm$ sign!).
4. Rearrange to get x by itself.

COMMON MISTAKE: When taking the square root of both sides, NEVER forget the $\pm$ sign — omitting the negative root loses the final accuracy mark.

MODEL WORKED EXAMPLES

EXAMPLE 2.1 **Calculator**

By completing the square, solve $x^2 - 6x + 2 = 0$. Give your answers to 2 decimal places.

STEP-BY-STEP SOLUTION:

1. Complete the square on $x^2 - 6x + 2$:

$$(x - 3)^2 - 9 + 2 = 0 \implies (x - 3)^2 - 7 = 0$$

2. Move the -7 over:

$$(x - 3)^2 = 7$$

3. Square root both sides:

$$x - 3 = \pm\sqrt{7}$$

4. Add 3 to both sides:

$$x = 3 \pm \sqrt{7}$$

Calculate the two values:

$$x = 0.35, \text{ or } x = 5.65$$

EXAMPLE 2.2 **Non-Calculator**

By completing the square, solve $2x^2 + 16x + 18 = 0$. Give your answers in surd form.

STEP-BY-STEP SOLUTION:

1. Factor out 2 from the x terms and complete the square:

$$2[(x + 4)^2 - 16] + 18 = 0 \implies 2(x + 4)^2 - 32 + 18 = 0$$

$$2(x + 4)^2 - 14 = 0$$

2. Move the -14 over and divide by 2:

$$2(x + 4)^2 = 14 \implies (x + 4)^2 = 7$$

3. Square root both sides:

$$x + 4 = \pm\sqrt{7}$$

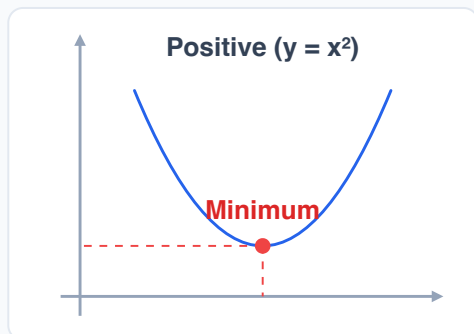
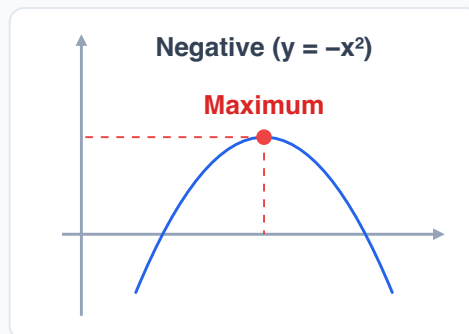
4. Subtract 4 from both sides:

$$x = -4 \pm \sqrt{7}$$

3 Turning Points

□ CORE CONCEPT & METHOD

Completing the square reveals the **turning point** (the lowest or highest tip) of a quadratic curve!



Once your quadratic is written in completed square form $(x + a)^2 + b$, the turning point is at the coordinate $(-a, b)$.

Just **flip the sign** of the number inside the bracket for the x -coordinate, and keep the outside number exactly the same for the y -coordinate!

COMMON MISTAKE: For $y = a(x + p)^2 + q$, flip ONLY the sign inside the bracket to get $(-p, q)$, and remember $a < 0$ gives a MAXIMUM point, not a minimum.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

✓ Non-Calculator

By completing the square, find the coordinates of the minimum point of the curve $y = x^2 - 12x + 40$.

STEP-BY-STEP SOLUTION:

1. Complete the square on $x^2 - 12x + 40$:

$$y = (x - 6)^2 - 36 + 40 = (x - 6)^2 + 4$$

2. Look inside the bracket: -6 . Flip the sign to get $+6$. This is the x -coordinate.

3. Look outside the bracket: $+4$. Keep it the same. This is the y -coordinate.

Put them together as a coordinate pair (x, y) :

$$(6, 4)$$

EXAMPLE 3.2 **Non-Calculator**

By completing the square, find the coordinates of the maximum point of the curve $y = 6 + 8x - x^2$.

STEP-BY-STEP SOLUTION:

1. Factor out -1 from the x terms:

$$y = -(x^2 - 8x) + 6$$

2. Complete the square inside the bracket:

$$y = -[(x - 4)^2 - 16] + 6 = -(x - 4)^2 + 16 + 6 = -(x - 4)^2 + 22$$

3. Look inside the bracket: -4 . Flip the sign to get $+4$.

4. Look outside the bracket: $+22$. Keep it the same.

Put them together:

$$(4, 22)$$

Quadratic Simultaneous Equations

CORE CONCEPT & METHOD

When one equation has a squared term (like x^2 or y^2), you cannot use the elimination method. Instead, you must use **Substitution!**

- **Step 1:** Rearrange the linear (simple) equation to get $x =$ or $y =$.
- **Step 2:** Substitute this expression into the quadratic equation.
- **Step 3:** Expand, simplify, and set to 0 to solve the quadratic.
- **Step 4:** Substitute both answers back into the linear equation to find the pairs!

COMMON MISTAKE: Always substitute the linear expression inside brackets — e.g.
 $(7 - 3x)^2 = 49 - 42x + 9x^2$, NEVER $49 + 9x^2$ — and pair each x with its matching y .

MODEL WORKED EXAMPLES

EXAMPLE 1

 Non-Calculator

Solve the simultaneous equations:

$$x^2 + y^2 = 25$$

$$x = y - 7$$

STEP-BY-STEP SOLUTION:

1. The linear equation is already arranged as $x = y - 7$.We substitute $(y - 7)$ in place of x in the quadratic equation:

$$(y - 7)^2 + y^2 = 25$$

2. Expand the bracket.

(Remember $(y - 7)^2 = (y - 7)(y - 7) = y^2 - 14y + 49$):

$$y^2 - 14y + 49 + y^2 = 25$$

$$2y^2 - 14y + 49 = 25$$

3. Set to 0:

$$2y^2 - 14y + 49 = 25 \quad \underline{-25} \quad -25$$

$$2y^2 - 14y + 24 = 0$$

4. Divide by 2 to make it easier to factorise:

$$2y^2 - 14y + 24 = 0$$

$$\div 2 \quad \div 2$$

$$y^2 - 7y + 12 = 0$$

5. Factorise. Two numbers that multiply to $+12$ and add to -7 are -3 and -4 :

$$(y - 3)(y - 4) = 0$$

So $y = 3$ or $y = 4$.6. Substitute back into the linear equation ($x = y - 7$):When $y = 3$:

$$x = 3 - 7 = -4$$

When $y = 4$:

$$x = 4 - 7 = -3$$

$$x = -4, y = 3 \text{ and } x = -3, y = 4$$

EXAMPLE 2 **Non-Calculator**

Solve algebraically the simultaneous equations:

$$x^2 - 3y^2 = 24$$

$$2x + 3y = 6$$

STEP-BY-STEP SOLUTION:1. Rearrange the linear equation. Let's make x the subject:

$$\begin{array}{r} 2x + 3y = 6 \\ \underline{-3y} \quad -3y \\ 2x = 6 - 3y \end{array}$$

$$x = \frac{6 - 3y}{2}$$

2. Substitute this expression in place of x in the quadratic equation:

$$\left(\frac{6 - 3y}{2}\right)^2 - 3y^2 = 24$$

3. Expand the bracket. Square the top and square the bottom:

$$\frac{(6 - 3y)(6 - 3y)}{4} - 3y^2 = 24$$

$$\frac{36 - 36y + 9y^2}{4} - 3y^2 = 24$$

4. Multiply everything by 4 to remove the fraction:

$$36 - 36y + 9y^2 - 12y^2 = 96$$

$$-3y^2 - 36y + 36 = 96$$

5. Set to 0:

$$\begin{array}{r} -3y^2 - 36y + 36 = 96 \\ \underline{-96 \quad -96} \end{array}$$

$$-3y^2 - 36y - 60 = 0$$

6. Divide by -3 to make it a positive y^2 :

$$\begin{array}{r} -3y^2 - 36y - 60 = 0 \\ \underline{\div -3 \quad \div -3} \end{array}$$

$$y^2 + 12y + 20 = 0$$

7. Factorise:

$$(y + 10)(y + 2) = 0$$

So $y = -10$ or $y = -2$.8. Substitute back into $x = \frac{6 - 3y}{2}$:When $y = -10$:

$$x = \frac{6 - 3(-10)}{2} = \frac{36}{2} = 18$$

When $y = -2$:

$$x = \frac{6 - 3(-2)}{2} = \frac{12}{2} = 6$$

$$x = 6, y = -2 \text{ and } x = 18, y = -10$$

Quadratic Inequalities

CORE CONCEPT & METHOD

To solve a quadratic inequality (like $x^2 + 5x + 6 > 0$):

- **Step 1:** Factorise the quadratic (or use the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ if it does not factorise) to find the *critical values* where it equals 0.
- **Step 2:** Sketch a quick U-shaped parabola crossing the x -axis at those critical values.
- **Step 3:** Look at the inequality sign. If it is > 0 , you want the parts of the graph *above* the axis (the two separate outer tails). If it is < 0 , you want the part *below* the axis (the single connected region in the middle).

COMMON MISTAKE: Never write a split outer region like $x < -3$ or $x > 5$ as a single combined statement $-3 > x > 5$ — Edexcel awards zero final marks for invalid inequality notation.

MODEL WORKED EXAMPLES

EXAMPLE 1

 **Non-Calculator**

Solve the inequality $x^2 - 3x - 10 > 0$

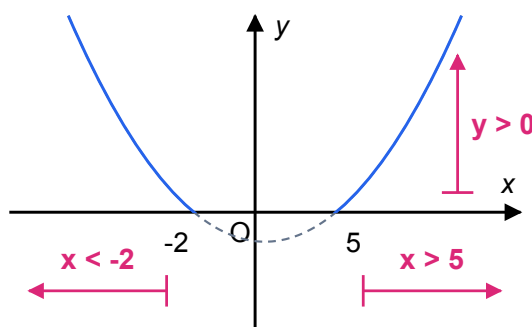
STEP-BY-STEP SOLUTION:

1. Factorise to find the critical values:

$$(x - 5)(x + 2) = 0$$

The critical values are $x = 5$ and $x = -2$.

2. Think of a U-shaped graph crossing at -2 and 5 :



3. The inequality is > 0 , which means we want the parts of the graph ABOVE the x -axis. These are the two outer tails going outwards in opposite directions.

So $x < -2$ or $x > 5$.

$$x < -2 \text{ or } x > 5$$

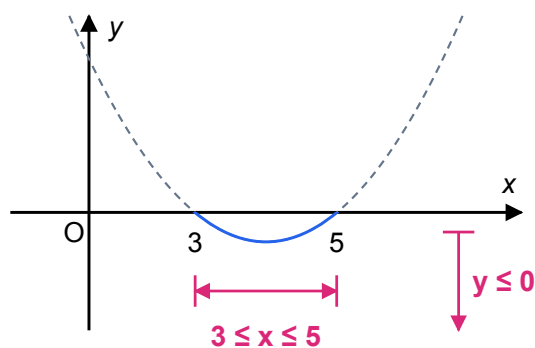
EXAMPLE 2**Non-Calculator**Solve the inequality $x^2 - 8x + 15 \leq 0$ **STEP-BY-STEP SOLUTION:**

1. Factorise to find the critical values:

$$(x - 5)(x - 3) = 0$$

The critical values are $x = 3$ and $x = 5$.

2. Think of a U-shaped graph crossing at 3 and 5:

3. The inequality is ≤ 0 , which means we want the part of the graph BELOW the x-axis. This is the single continuous dip connected between 3 and 5.So $3 \leq x \leq 5$.

$$3 \leq x \leq 5$$

Iteration

1 Using Iteration Formulae

□ CORE CONCEPT & METHOD

What is Iteration?

Iteration means repeating a process. In maths, we take an initial value (usually called x_0) and plug it into a formula to get the next value, x_1 . We then plug x_1 back into the formula to get x_2 , and so on.

Calculator Tip:

Type your x_0 value and press $=$. Then, type the formula using the **ANS** button instead of x_n . Every time you press $=$, you get the next term in the sequence!

COMMON MISTAKE: Use your calculator **ANS** key with the **FULL** unrounded display at each step — rounding x_1 or x_2 early causes drift in the 3rd decimal place of x_3 .

MODEL WORKED EXAMPLES

EXAMPLE 1.1

 **Calculator**

Using $x_{n+1} = 2 + \frac{8}{x_n^2}$ with $x_0 = 2$. Find the value of x_1, x_2, x_3 .

STEP-BY-STEP SOLUTION:

$$x_1 = 2 + \frac{8}{2^2} = 2 + 2 = 4$$

$$x_2 = 2 + \frac{8}{4^2} = 2 + 0.5 = 2.5$$

$$x_3 = 2 + \frac{8}{2.5^2} = 2 + 1.28 = 3.28$$

2 Estimating Roots & Rearranging

□ CORE CONCEPT & METHOD

Rearranging into an Iteration Formula

Iteration is a powerful tool for solving complex equations (like cubics) that don't easily factorise. To do this, we rearrange the equation to make x the subject on one side, and then turn it into an iteration formula.

Finding the original equation:

If you are given $x_{n+1} = \frac{5}{x_n^2} + 2$, you can find the original equation by replacing x_{n+1} and x_n with just x , and rearranging to equal zero:

$$x = \frac{5}{x^2} + 2$$

$$x - 2 = \frac{5}{x^2}$$

$$x^3 - 2x^2 = 5$$

$$x^3 - 2x^2 - 5 = 0$$

COMMON MISTAKE: Computing $f(a)$ and $f(b)$ alone is NOT enough — you must explicitly state "change of sign (and $f(x)$ is continuous), so a root lies between a and b ".

MODEL WORKED EXAMPLES

EXAMPLE 2.1

✎ Non-Calculator

Show that $x^3 - 3x^2 - 2 = 0$ has a root (or solution) between $x = 3$ and $x = 4$.

STEP-BY-STEP SOLUTION:

$$\text{Let } f(x) = x^3 - 3x^2 - 2$$

$$f(3) = 3^3 - 3(3)^2 - 2 = 27 - 27 - 2 = -2$$

$$f(4) = 4^3 - 3(4)^2 - 2 = 64 - 48 - 2 = 14$$

Because there is a **change of sign** between $f(3)$ (negative) and $f(4)$ (positive), a root (or solution) must exist between 3 and 4.

EXAMPLE 2.2 **Non-Calculator**

Show that the equation $4x^3 - x^2 - 7 = 0$ can be rearranged to give $x = \sqrt{\frac{7}{4x-1}}$.

STEP-BY-STEP SOLUTION:

1. Add 7 to both sides to isolate the x terms:

$$4x^3 - x^2 = 7$$

2. Factorise x^2 out of the left-hand side:

$$x^2(4x - 1) = 7$$

3. Divide both sides by $(4x - 1)$:

$$x^2 = \frac{7}{4x - 1}$$

4. Square root both sides:

$$x = \sqrt{\frac{7}{4x - 1}}$$

Transforming Graphs

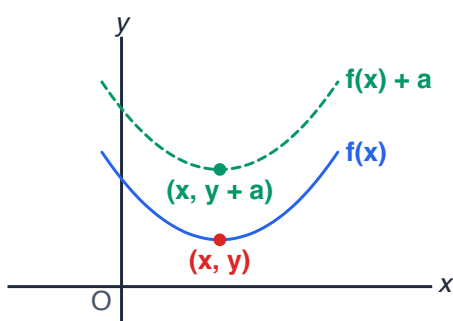
1 Translations of Graphs: $y = f(x) \pm a$ and $y = f(x \pm a)$

□ CORE CONCEPT & METHOD

Translations of Graphs: $y = f(x) \pm a$ and $y = f(x \pm a)$

Adding or subtracting a constant shifts (translates) the entire curve without changing its shape or orientation.

1. Outside the Bracket: $y = f(x) \pm a$ (Vertical)



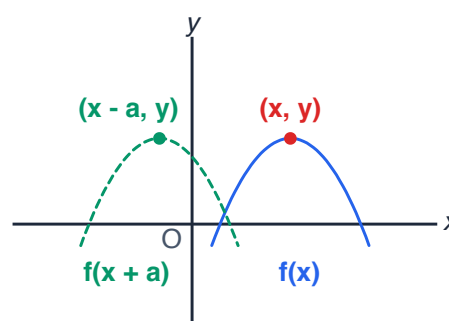
- Affects the **y -coordinates only** ("does what it says"):

$$y = f(x) + a \text{ shifts UP by } a: \begin{pmatrix} 0 \\ a \end{pmatrix}$$

$$y = f(x) - a \text{ shifts DOWN by } a: \begin{pmatrix} 0 \\ -a \end{pmatrix}$$

- Rule: $(x, y) \rightarrow (x, y \pm a)$.

2. Inside the Bracket: $y = f(x \pm a)$ (Horizontal)



- Affects the **x -coordinates in the opposite direction**:

$$y = f(x + a) \text{ shifts LEFT by } a: \begin{pmatrix} -a \\ 0 \end{pmatrix}$$

$$y = f(x - a) \text{ shifts RIGHT by } a: \begin{pmatrix} a \\ 0 \end{pmatrix}$$

- Rule: $(x, y) \rightarrow (x \mp a, y)$.

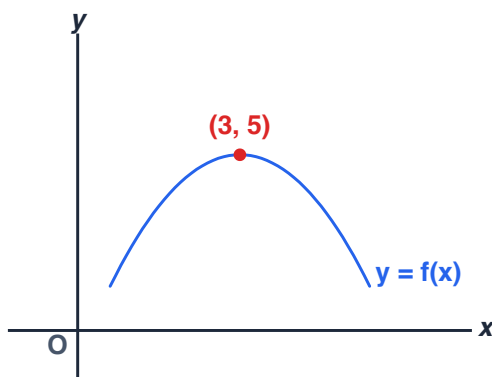
COMMON MISTAKE: Changes INSIDE $f(x + a)$ shift horizontally the OPPOSITE way to the sign (left by a , leaving y unchanged), whereas $f(x) + a$ shifts vertically (x unchanged).

MODEL WORKED EXAMPLES

EXAMPLE 1.1

Non-Calculator

The diagram shows the curve $y = f(x)$ with a maximum point at $(3, 5)$.



Write down the coordinates of the maximum point on:

(a) $y = f(x) - 4$

(b) $y = f(x + 2)$

STEP-BY-STEP SOLUTION:

1. For $y = f(x) - 4$, the -4 is **outside** the bracket so it translates the curve 4 units downwards:

$$(3, 5) \rightarrow (3, 5 - 4) = (3, 1)$$

2. For $y = f(x + 2)$, the $+2$ is **inside** the bracket so it translates the curve 2 units to the left:

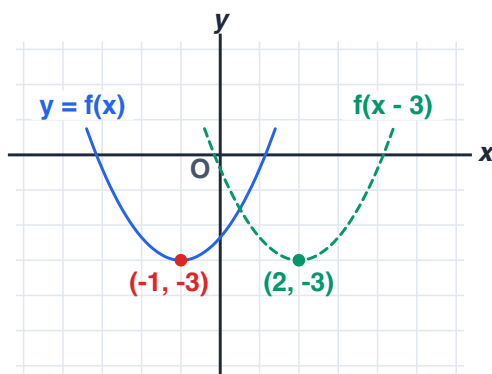
$$(3, 5) \rightarrow (3 - 2, 5) = (1, 5)$$

$$(a) (3, 1), \quad (b) (1, 5)$$

EXAMPLE 1.2

Non-Calculator

The curve $y = f(x)$ shown on the grid has a minimum turning point at $(-1, -3)$.



Find the coordinates of the minimum turning point of $y = f(x - 3)$.

STEP-BY-STEP SOLUTION:

1. The -3 is inside the bracket, which translates the graph 3 units to the **right** (add 3 to the x -coordinate).

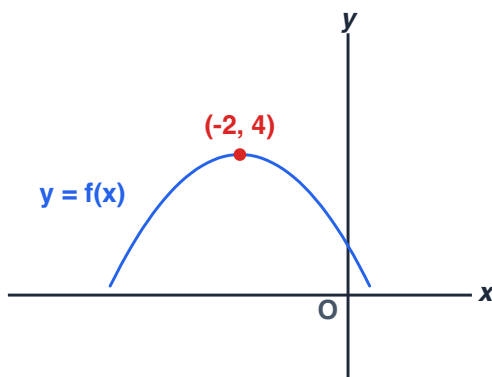
2. Adding 3 to the x -coordinate gives:

$$(-1 + 3, -3) = (2, -3)$$

$$(2, -3)$$

EXAMPLE 1.3 **Non-Calculator**

The curve $y = f(x)$ has a turning point at $(-2, 4)$.



Find the coordinates of the turning point of $y = f(x - 2) + 5$.

STEP-BY-STEP SOLUTION:

1. Inside the bracket, $x - 2$ translates the curve 2 units to the right: $x = -2 + 2 = 0$.
2. Outside the bracket, $+5$ translates the curve 5 units upwards: $y = 4 + 5 = 9$.
3. The new turning point is $(0, 9)$.

$(0, 9)$

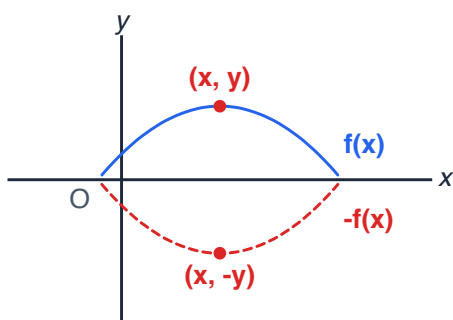
2 Reflections of Graphs: $y = -f(x)$ and $y = f(-x)$

□ CORE CONCEPT & METHOD

Reflections of Graphs: $y = -f(x)$ and $y = f(-x)$

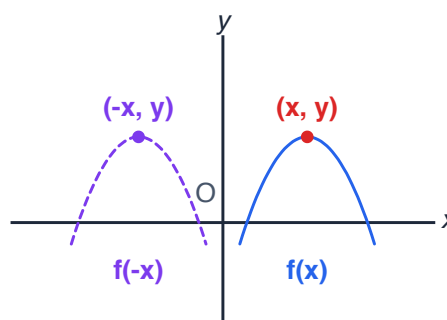
A negative sign outside or inside the function reflects the graph across one of the coordinate axes.

1. Outside the Bracket: $y = -f(x)$ (Reflect in x -axis)



- Multiplies every **y -coordinate** by -1 .
- Reflection in the **x -axis** ($y = 0$).
- Rule: $(x, y) \rightarrow (x, -y)$ (a maximum point becomes a minimum point!).

2. Inside the Bracket: $y = f(-x)$ (Reflect in y -axis)



- Replaces x with $-x$, multiplying every **x -coordinate** by -1 .
- Reflection in the **y -axis** ($x = 0$).
- Rule: $(x, y) \rightarrow (-x, y)$.

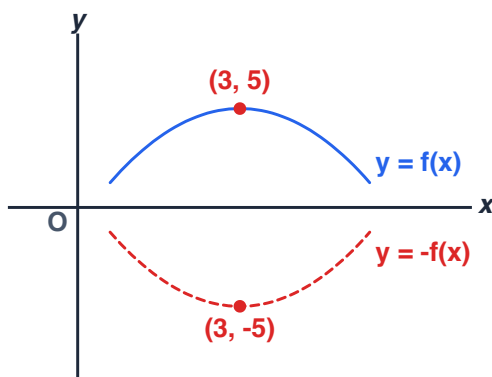
COMMON MISTAKE: Do not mix up the axes: $y = -f(x)$ negates y -coordinates (reflection in the x -axis), whereas $y = f(-x)$ negates x -coordinates (reflection in the y -axis).

MODEL WORKED EXAMPLES

EXAMPLE 2.1

Non-Calculator

The diagram shows the curve $y = f(x)$ with a maximum point at $(3, 5)$.



Write down the coordinates of the turning point of the curve $y = -f(x)$.

STEP-BY-STEP SOLUTION:

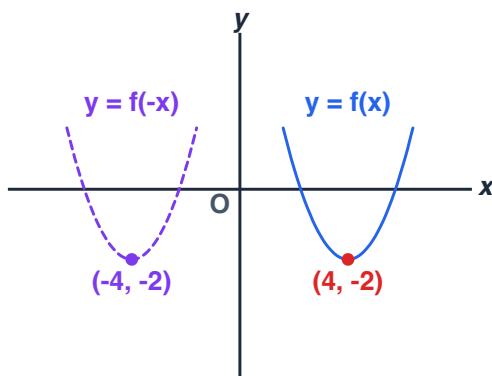
1. The negative sign is **outside** the bracket, which reflects the graph in the x -axis.
2. Under a reflection in the x -axis, $(x, y) \rightarrow (x, -y)$, so the maximum at $(3, 5)$ becomes a minimum at $(3, -5)$.

$(3, -5)$

EXAMPLE 2.2

Non-Calculator

The diagram shows the curve $y = f(x)$ with a minimum point at $(4, -2)$.



Write down the coordinates of the minimum point of the curve $y = f(-x)$.

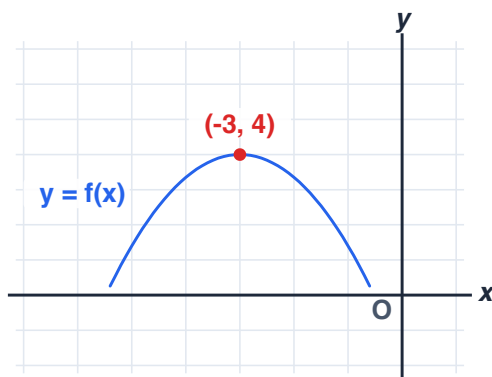
STEP-BY-STEP SOLUTION:

1. The negative sign is **inside** the bracket, which reflects the graph in the y -axis.
2. Under a reflection in the y -axis, $(x, y) \rightarrow (-x, y)$, so $(4, -2)$ moves to $(-4, -2)$.

$(-4, -2)$

EXAMPLE 2.3 **Non-Calculator**

The curve $y = f(x)$ on the grid has a turning point at $(-3, 4)$.



Find the coordinates of the turning point on:

(a) $y = -f(x)$

(b) $y = f(-x)$

STEP-BY-STEP SOLUTION:

1. For $y = -f(x)$, reflect in the x -axis (negate y): $(-3, 4) \rightarrow (-3, -4)$.
2. For $y = f(-x)$, reflect in the y -axis (negate x): $(-3, 4) \rightarrow (3, 4)$.

$$(a) (-3, -4), \quad (b) (3, 4)$$

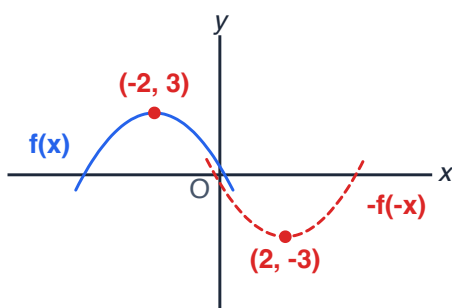
3 Combined Transformations & Identifying Equations

□ CORE CONCEPT & METHOD

Combined Transformations & Identifying Equations

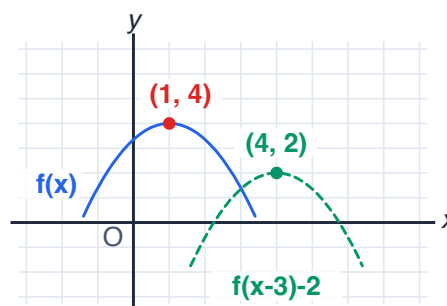
When a function has both an inside transformation (affecting x) and an outside transformation (affecting y), apply the x -change to the x -coordinate and the y -change to the y -coordinate independently!

1. Combined Coordinate Mapping



- $y = -f(x + a): (x, y) \rightarrow (x - a, -y)$
- $y = f(-x) + b: (x, y) \rightarrow (-x, y + b)$
- $y = -f(x) + c: (x, y) \rightarrow (x, -y + c)$

2. Identifying a Transformation from a Grid



Compare the turning point (x_1, y_1) of $y = f(x)$ with (x_2, y_2) on the transformed curve:

- Same way up & shifted by $\begin{pmatrix} h \\ k \end{pmatrix} \rightarrow$
 $y = f(x - h) + k.$

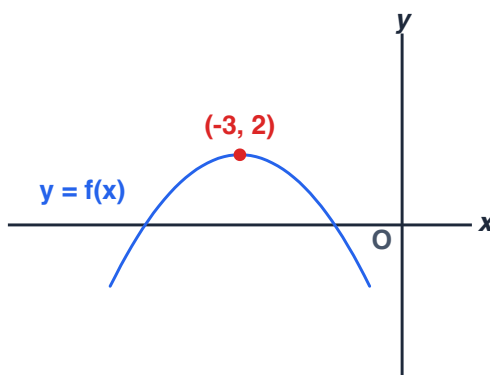
COMMON MISTAKE: When translating an algebraic curve like $y = x^2 - 6x + 11$ by $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$, replace EVERY x with $(x + 2)$ inside brackets before expanding and adding 3.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

Non-Calculator

Shown below is the curve $y = f(x)$ with a maximum turning point at $(-3, 2)$.



Write down the coordinates of the turning point of the curve with equation $y = -f(x + 4)$.

STEP-BY-STEP SOLUTION:

1. Inside the bracket, $x + 4$ shifts the x -coordinate 4 units to the left:

$$x = -3 - 4 = -7$$

2. Outside the bracket, the negative sign reflects the curve in the x -axis (negating the y -coordinate):

$$y = -(2) = -2$$

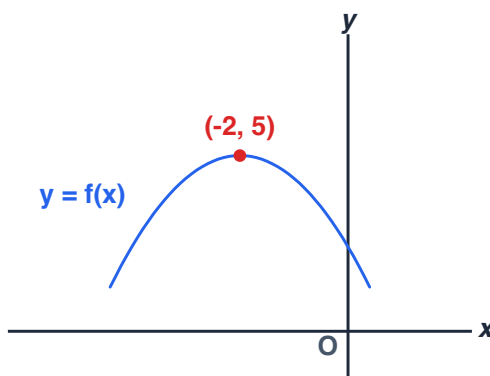
3. Therefore, the new turning point is $(-7, -2)$.

$$(-7, -2)$$

EXAMPLE 3.2

Non-Calculator

The curve $y = f(x)$ has a turning point at $(-2, 5)$.



Find the coordinates of the turning point of $y = f(-x) - 3$.

STEP-BY-STEP SOLUTION:

1. Inside the bracket, $-x$ reflects the curve in the y -axis (negating the x -coordinate):

$$x = -(-2) = 2$$

2. Outside the bracket, -3 translates the curve 3 units downwards:

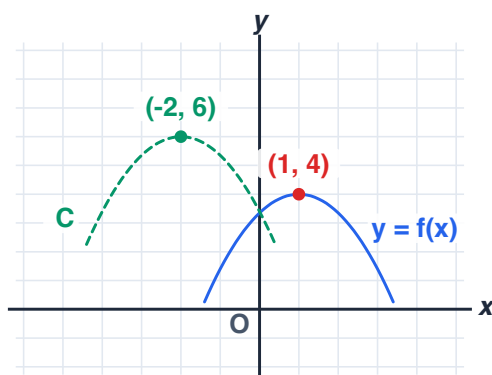
$$y = 5 - 3 = 2$$

3. Therefore, the new turning point is $(2, 2)$.

$$(2, 2)$$

EXAMPLE 3.3 **Non-Calculator**

The grid shows the curve $y = f(x)$ with maximum point $(1, 4)$ and a transformed curve C with maximum point $(-2, 6)$.



Write down the equation of curve C in the form $y = f(x + a) + b$.

STEP-BY-STEP SOLUTION:

1. Compare the maximum points: $(1, 4) \rightarrow (-2, 6)$ is a shift of 3 units left (-3 in x) and 2 units up ($+2$ in y).
2. A horizontal shift of 3 units left is written inside the bracket as $f(x + 3)$.
3. A vertical shift of 2 units up is written outside the bracket as $+2$, giving $y = f(x + 3) + 2$.

$$y = f(x + 3) + 2$$

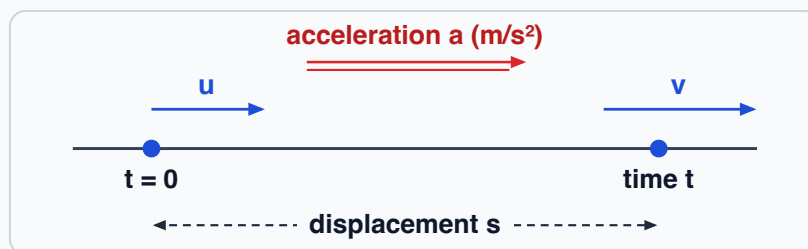
Kinematics

1 Constant Acceleration Formulae (SUVAT)

□ CORE CONCEPT & METHOD

When an object moves in a straight line with **constant acceleration**, five variables link its motion:

- s = **displacement** (distance in a given direction, in m)
- u = **initial velocity** (m/s) — note $u = 0$ if starting *from rest*
- v = **final velocity** (m/s) — note $v = 0$ if coming *to rest*
- a = **constant acceleration** (m/s²) — **negative** if decelerating!
- t = **time taken** (s)



Three Core SUVAT Equations (choose the one missing the unneeded variable):

$$1. v = u + at$$

(no s)

$$2. s = ut + \frac{1}{2}at^2$$

(no v)

$$3. v^2 = u^2 + 2as$$

(no t)

COMMON MISTAKE: If an object slows down (decelerates), substitute a as a **NEGATIVE** value into $v = u + at$ or $v^2 = u^2 + 2as$, and set $u = 0$ ("from rest") or $v = 0$ ("comes to rest").

MODEL WORKED EXAMPLES

EXAMPLE 1.1

 Non-Calculator

A car accelerates uniformly from $u = 6 \text{ m/s}$ at $a = 2.5 \text{ m/s}^2$ for $t = 8 \text{ s}$.

Find:

- (a) its final velocity v in m/s ,
 (b) the distance travelled s in m .

STEP-BY-STEP SOLUTION:

- (a) List knowns: $u = 6$, $a = 2.5$, $t = 8$. Use $v = u + at$:

$$v = 6 + (2.5)(8) = 6 + 20 = 26 \text{ m/s}$$

- (b) Use $s = ut + \frac{1}{2}at^2$:

$$s = (6)(8) + \frac{1}{2}(2.5)(8^2) = 48 + 80 = 128 \text{ m}$$

(a) 26 m/s, (b) 128 m

EXAMPLE 1.2

 Calculator

A train travelling at 28 m/s brakes with a constant deceleration of 1.5 m/s^2 until its speed is 16 m/s .

Find the distance travelled while braking, in metres.

STEP-BY-STEP SOLUTION:

1. Identify the known SUVAT values: $u = 28$, $v = 16$, and $a = -1.5$ (negative because the train is decelerating).

2. Since time t is not involved, substitute into $v^2 = u^2 + 2as$:

$$\begin{aligned} 16^2 &= 28^2 + 2(-1.5)s \\ 256 &= 784 - 3s \\ 3s &= 528 \implies s = 176 \text{ m} \end{aligned}$$

176 m

2 Algebraic Multi-Stage Journey Problems

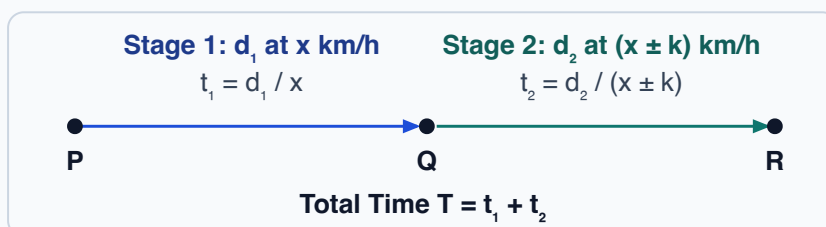
□ CORE CONCEPT & METHOD

In Higher GCSE multi-stage journey problems, divide the trip into clear stages and link them through **total distance** and **total time**:

- Average Speed = $\frac{\text{Total Distance}}{\text{Total Time}}$ (never simply average the two stage speeds!).
- When stage speeds are algebraic (e.g. x km/h and $(x \pm k)$ km/h), write each stage time using $t = \frac{d}{s}$ and set their sum equal to the total time T :

$$\frac{d_1}{x} + \frac{d_2}{x \pm k} = T$$

- Multiply every term by $x(x \pm k)$ to clear the fractions, rearrange into $ax^2 + bx + c = 0$, and solve (rejecting any negative or invalid speed).



COMMON MISTAKE: NEVER average the two stage speeds — always use
 Average Speed = Total Distance $\div$ Total Time, finding each stage time separately via $t = d \div s$.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

A cyclist rides 24 km at a constant speed of x km/h and then rides a further 14 km at a constant speed of $(x + 2)$ km/h.

The total time for the whole 38 km journey is 3 hours.

Form and solve a quadratic equation to find the value of x .

STEP-BY-STEP SOLUTION:

1. Write the time for each stage using $\text{Time} = \frac{\text{Distance}}{\text{Speed}}$:

$$\frac{24}{x} + \frac{14}{x+2} = 3$$

2. Multiply every term by $x(x + 2)$ to clear the denominators:

$$\begin{aligned} 24(x + 2) + 14x &= 3x(x + 2) \\ 38x + 48 &= 3x^2 + 6x \implies 3x^2 - 32x - 48 = 0 \end{aligned}$$

3. Factorise and solve:

$$(3x + 4)(x - 12) = 0$$

Since speed $x > 0$, reject $x = -\frac{4}{3}$, so $x = 12$ km/h.

$$x = 12$$

Velocity-Time Graphs

CORE CONCEPT & METHOD

On a velocity-time graph:

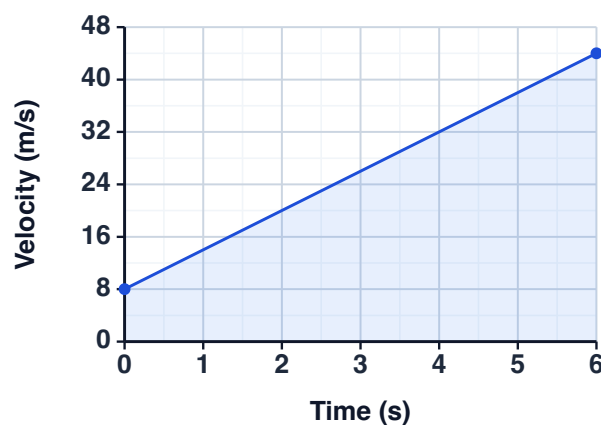
- **Acceleration** is the **GRADIENT** ($\frac{\text{change in velocity}}{\text{change in time}}$).
- **Distance travelled** is the **AREA under the graph** (split into triangles, rectangles, or trapeziums).

COMMON MISTAKE: On a velocity-time graph, Distance is the AREA under the graph (not peak velocity $\times$ total time), and read the numbered axis scales rather than counting squares.

MODEL WORKED EXAMPLES

EXAMPLE 1

 **Non-Calculator**



A velocity-time graph shows a straight line from (0, 8) to (6, 44).

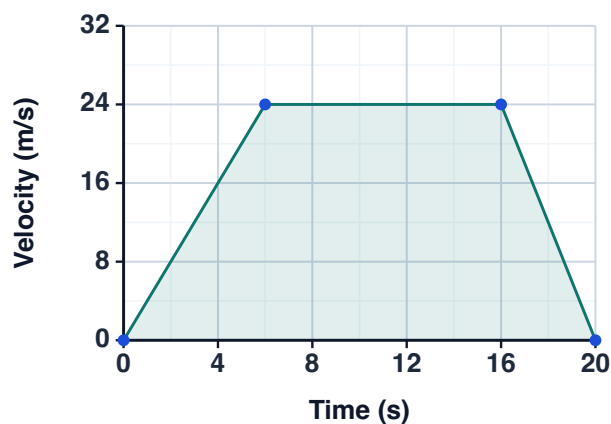
- (a) Calculate the acceleration.
 (b) Calculate the total distance travelled.

STEP-BY-STEP SOLUTION:

(a) Acceleration = gradient = $\frac{44 - 8}{6 - 0} = \frac{36}{6} = 6 \text{ m/s}^2$.

- (b) Distance = area under graph (a trapezium with parallel vertical sides 8 and 44, width 6):

$$d = \frac{1}{2}(8 + 44) \times 6 = \frac{1}{2} \times 52 \times 6 = 156 \text{ m}$$

EXAMPLE 2 **Non-Calculator**

The velocity-time graph shows a 20-second journey.

- (a) Find the acceleration in the first 6 seconds.
 (b) Find the total distance travelled over the 20 seconds.

STEP-BY-STEP SOLUTION:

(a) Acceleration = $\frac{24 - 0}{6 - 0} = 4 \text{ m/s}^2$.

- (b) Total distance = area of trapezium with base 20 s, top width $16 - 6 = 10$ s, and height 24 m/s:

$$d = \frac{1}{2}(20 + 10) \times 24 = 360 \text{ m}$$

ALGEBRA — ALGEBRAIC PROOF

Algebraic Proof

1 Setting up Proofs

□ CORE CONCEPT & METHOD

To prove something algebraically, you must use letters instead of numbers.

- Any integer: n
- An even number: $2n$
- An odd number: $2n + 1$
- Consecutive integers: $n, n + 1, n + 2$
- Consecutive even numbers: $2n, 2n + 2, 2n + 4$

COMMON MISTAKE: Never use $2n$ and $2m$ for consecutive even integers — link both to the SAME integer n (use $2n$ and $2n + 2$ for evens, or $2n + 1$ and $2n + 3$ for consecutive odds).

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✎ Non-Calculator

Prove algebraically that the sum of any five consecutive integers is always a multiple of 5.

STEP-BY-STEP SOLUTION:

Let the five consecutive integers be $n, n + 1, n + 2, n + 3$, and $n + 4$.

$$\text{Sum} = n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10$$

Factorise: $5(n + 2)$

Since it is a multiple of 5, the statement is proven.

2 Expanding to Prove

□ CORE CONCEPT & METHOD

Most algebraic proofs just require you to **expand all brackets and simplify**.

If you are asked to prove an expression is a multiple of 12, your final simplified expression will usually look like $12n$.

COMMON MISTAKE: Do not stop at $8n^2 + 8n + 4$ — to prove a multiple of k , you MUST factor k out explicitly as $4(2n^2 + 2n + 1)$ and write a brief concluding statement.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

✎ Non-Calculator

Prove that $(n + 3)^2 - (n - 3)^2$ is always a multiple of 12.

STEP-BY-STEP SOLUTION:

Expand both brackets:

$$(n^2 + 6n + 9) - (n^2 - 6n + 9)$$

Simplify:

$$n^2 + 6n + 9 - n^2 + 6n - 9 = 12n$$

Since $12n$ is a multiple of 12 for any integer n , the statement is proven.

Equation of a Circle

1 Centre and Radius

□ CORE CONCEPT & METHOD

The equation of a circle centered at the origin $(0, 0)$ is $x^2 + y^2 = r^2$, where r is the radius.

Remember to **square root** the number to find r !

COMMON MISTAKE: In $x^2 + y^2 = r^2$, the constant on the right is r^2 , NOT r — always take the positive square root to find the radius (e.g. $x^2 + y^2 = 225 \implies r = 15$).

MODEL WORKED EXAMPLES

EXAMPLE 1.1

✓ Non-Calculator

A circle has the equation $x^2 + y^2 = 81$. Find the radius.

STEP-BY-STEP SOLUTION:

$$r^2 = 81, \text{ so } r = \sqrt{81} = 9.$$

9

2 Points on a Circle

□ CORE CONCEPT & METHOD

To check if a point (x, y) lies on the circle $x^2 + y^2 = r^2$, **substitute** the coordinates into the equation and check if it equals r^2 .

COMMON MISTAKE: Always wrap negative coordinates in brackets before squaring:
 $(-8)^2 + 15^2 = +64 + 225 = 289$, NEVER $-64 + 225$.

MODEL WORKED EXAMPLES

EXAMPLE 2.1

✓ Non-Calculator

Show that $(6, 8)$ lies on $x^2 + y^2 = 100$.

STEP-BY-STEP SOLUTION:

$$6^2 + 8^2 = 36 + 64 = 100. \text{ This equals the radius squared, so it lies on the circle.}$$

100

3 Area and Circumference

□ CORE CONCEPT & METHOD

Once you know r^2 , you can easily find the area (πr^2)! You don't even need to square root it first.

COMMON MISTAKE: Check whether the question asks for exact AREA (πr^2 , using r^2 directly) or CIRCUMFERENCE ($2\pi r$, square-rooting r^2 first), and leave π in your answer.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

✓ Non-Calculator

Find the exact area of the circle $x^2 + y^2 = 49$.

STEP-BY-STEP SOLUTION:

Here, $r^2 = 49$.

Area = $\pi r^2 = 49\pi$.

$$49\pi$$

4 Translations

□ CORE CONCEPT & METHOD

If a circle centered at $(0, 0)$ is translated by a vector $\begin{pmatrix} a \\ b \end{pmatrix}$, the new centre becomes (a, b) .

COMMON MISTAKE: Translating a circle moves its CENTRE $(0, 0) \rightarrow (a, b)$ without changing its radius r , and inside $(x - a)^2 + (y - b)^2 = r^2$ the signs are OPPOSITE to (a, b) .

MODEL WORKED EXAMPLES

EXAMPLE 4.1

✓ Non-Calculator

A circle centered at $(0, 0)$ is translated by $\begin{pmatrix} -4 \\ 7 \end{pmatrix}$. What is its new centre?

STEP-BY-STEP SOLUTION:

The centre moves 4 units left and 7 units up.

New centre: $(-4, 7)$.

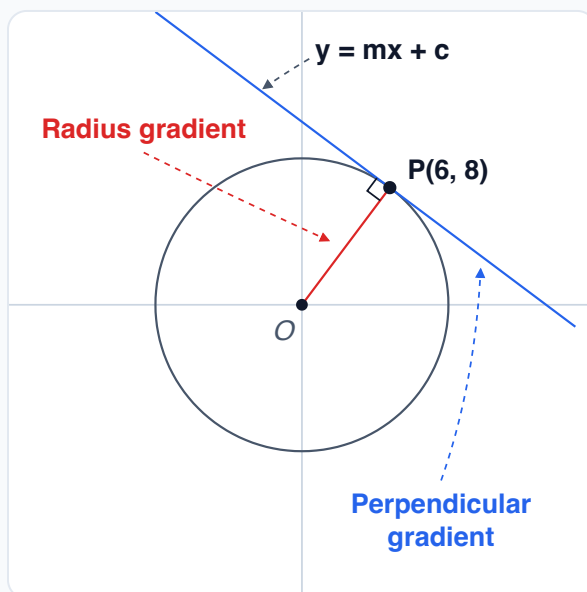
$$(-4, 7)$$

5 Tangents to Circles

□ CORE CONCEPT & METHOD

To find the **equation of a tangent**, follow 3 steps:

1. Find the gradient of the radius (from the origin to the point).
2. Find the perpendicular gradient (negative reciprocal).
3. Substitute the coordinates into $y = mx + c$ to find c !



COMMON MISTAKE: A tangent is PERPENDICULAR to the radius — after finding $m_{\text{radius}} = \frac{y_1}{x_1}$, take the negative reciprocal $m_{\text{tangent}} = -\frac{x_1}{y_1}$ before substituting (x_1, y_1) .

MODEL WORKED EXAMPLES

EXAMPLE 5.1

 Non-CalculatorFind the equation of the tangent to the circle $x^2 + y^2 = 100$ at the point $(6, 8)$.

STEP-BY-STEP SOLUTION:

1. **Radius gradient:** The centre is $(0, 0)$. The gradient to $(6, 8)$ is $\frac{8}{6} = \frac{4}{3}$.2. **Tangent gradient:** The negative reciprocal is $m = -\frac{3}{4}$.3. **Find c:** Substitute $x = 6, y = 8$ into $y = -\frac{3}{4}x + c$:

$$8 = -\frac{3}{4}(6) + c$$

$$8 = -\frac{9}{2} + c$$

$$c = 8 + \frac{9}{2} = \frac{25}{2}$$

$$\text{Equation: } y = -\frac{3}{4}x + \frac{25}{2}.$$

$$y = -\frac{3}{4}x + \frac{25}{2}$$

EXAMPLE 5.2

 Non-CalculatorFind the equation of the tangent to the circle $x^2 + y^2 = 289$ at the point $(-8, 15)$.

STEP-BY-STEP SOLUTION:

1. **Radius gradient:** The gradient to $(-8, 15)$ is $\frac{15}{-8} = -\frac{15}{8}$.2. **Tangent gradient:** The negative reciprocal is $m = \frac{8}{15}$.3. **Find c:** Substitute $x = -8, y = 15$ into $y = \frac{8}{15}x + c$:

$$15 = \frac{8}{15}(-8) + c$$

$$15 = -\frac{64}{15} + c$$

$$c = 15 + \frac{64}{15} = \frac{289}{15}$$

$$\text{Equation: } y = \frac{8}{15}x + \frac{289}{15}.$$

$$y = \frac{8}{15}x + \frac{289}{15}$$

Gradients & Rates of Change

1 Acceleration on Curves

▣ CORE CONCEPT & METHOD

If a velocity-time or distance-time graph is curved, the rate of change is constantly changing.

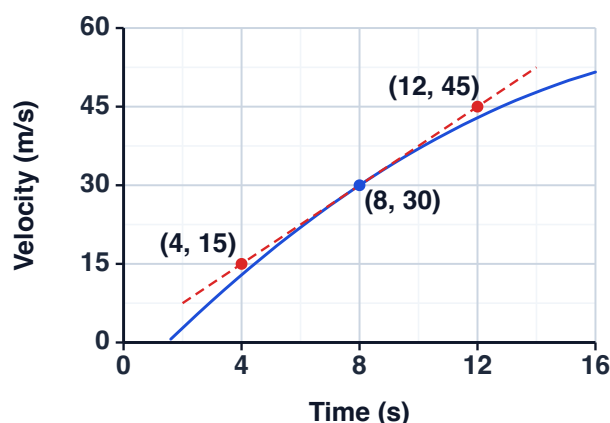
To find the **instantaneous rate of change** at a specific time t , use the **tangent line** touching the curve at that point and calculate its gradient $(\frac{y_2 - y_1}{x_2 - x_1})$.

COMMON MISTAKE: Find the gradient of the TANGENT using two widely spaced points (t_1, v_1) and (t_2, v_2) ON the tangent line — never divide $v \div t$ from a single point on the curve.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

☒ Non-Calculator



A curved velocity-time graph shows a car accelerating. A tangent is drawn at $t = 8$ s, passing through the points $(4, 15)$ and $(12, 45)$.

Find the instantaneous acceleration at $t = 8$ s.

STEP-BY-STEP SOLUTION:

The instantaneous acceleration is the **gradient of the tangent** at $t = 8$ s.

$$a = \frac{\text{change in } v}{\text{change in } t} = \frac{45 - 15}{12 - 4} = \frac{30}{8} = 3.75 \text{ m/s}^2$$

Note: Because the graph is curved, the tangent gives the exact *instantaneous* rate of change at $t = 8$ s.

2 Distance on Curves (Trapezium Rule)

▣ CORE CONCEPT & METHOD

To estimate the distance travelled under a curved velocity-time graph, we split the area into vertical strips of equal width h and treat them as **trapeziums**.

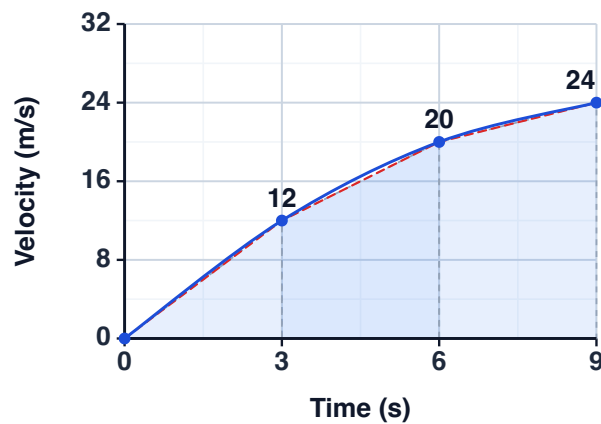
Area of each strip = $\frac{1}{2}(a + b)h$, where a and b are the vertical velocities at the left and right edges of the strip.

Overestimate or Underestimate? Compare the straight top chord of each trapezium with the curve: if the curve bends downwards (concave down, decreasing acceleration), the chords lie **below** the curve so the estimate is an **underestimate**; if the curve bends upwards (concave up, increasing acceleration), the chords lie **above** the curve so it is an **overestimate**.

COMMON MISTAKE: Count the STRIPS (gaps), not the vertical lines, to get strip width h , and justify under/overestimates geometrically ("chords lie below/above the curve").

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 Calculator


The curved velocity-time graph has been split into 3 strips of equal width $h = 3$ s. The velocities at each boundary are $v_0 = 0$, $v_1 = 12$, $v_2 = 20$, and $v_3 = 24$ m/s.

Use the trapezium rule to estimate the total distance travelled.

STEP-BY-STEP SOLUTION:

Area of each trapezium strip = $\frac{1}{2}(a + b) \times h$, where $h = 3$ s:

Strip 1: $\frac{1}{2}(0 + 12) \times 3 = 18$ m

Strip 2: $\frac{1}{2}(12 + 20) \times 3 = 48$ m

Strip 3: $\frac{1}{2}(20 + 24) \times 3 = 66$ m

$$\text{Total distance} \approx 18 + 48 + 66 = 132 \text{ m}$$

Vector Proofs

1 Vectors in Triangles

□ CORE CONCEPT & METHOD

To find a position vector like $\vec{OZ}$, first find the vector for the whole line (e.g., $\vec{XY}$). Then use the given ratio to find $\vec{XZ}$, and finally use $\vec{OZ} = \vec{OX} + \vec{XZ}$.

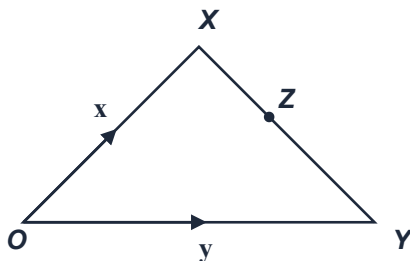
COMMON MISTAKE: Watch arrow directions: $\vec{AB} = \vec{OB} - \vec{OA} = -\vec{OA} + \vec{OB}$, and remember $AP : PB = 3 : 2$ means $AP = \frac{3}{5}AB$, NOT $\frac{3}{2}AB$.

MODEL WORKED EXAMPLES

EXAMPLE 1.1

 Non-Calculator

In triangle OXY , $\vec{OX} = \mathbf{x}$ and $\vec{OY} = \mathbf{y}$.
 Z is the point on XY such that $XZ : ZY = 3 : 4$.
 $\vec{OZ} = k(4\mathbf{x} + 3\mathbf{y})$. Find the value of k .



STEP-BY-STEP SOLUTION:

1. Find $\vec{XY}$:

$$\vec{XY} = -\mathbf{x} + \mathbf{y}$$

2. Find $\vec{XZ}$:

The ratio is $3 : 4$, so XZ is $\frac{3}{7}$ of the line.

$$\vec{XZ} = \frac{3}{7}(-\mathbf{x} + \mathbf{y}) = -\frac{3}{7}\mathbf{x} + \frac{3}{7}\mathbf{y}$$

3. Find $\vec{OZ}$:

$$\vec{OZ} = \vec{OX} + \vec{XZ}$$

$$\vec{OZ} = \mathbf{x} - \frac{3}{7}\mathbf{x} + \frac{3}{7}\mathbf{y} = \frac{4}{7}\mathbf{x} + \frac{3}{7}\mathbf{y}$$

4. Find k :

Factorise out $\frac{1}{7}$:

$$\vec{OZ} = \frac{1}{7}(4\mathbf{x} + 3\mathbf{y})$$

$$\text{So, } k = \frac{1}{7}.$$

$$\frac{1}{7}$$

2 Proving Collinearity

□ CORE CONCEPT & METHOD

To prove three points X , Y and Z are on the same straight line (collinear), you must prove that the vectors $\vec{XY}$ and $\vec{XZ}$ (or $\vec{YZ}$) are parallel. You do this by showing one is a **scalar multiple** of the other, e.g. $\vec{XZ} = 2\vec{XY}$.

COMMON MISTAKE: Showing $\vec{CD} = 4\vec{CM}$ only proves the vectors are PARALLEL — you MUST also write "and they share the common point C , so C, M, D are collinear".

MODEL WORKED EXAMPLES

EXAMPLE 2.1

Non-Calculator

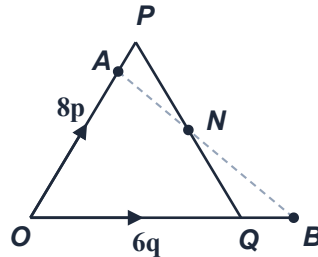
In the diagram, $\vec{OP} = 8\mathbf{p}$ and $\vec{OQ} = 6\mathbf{q}$.

A is the point on OP such that $OA : AP = 3 : 1$.

N is the midpoint of PQ .

B is the point on OQ extended such that $OQ : QB = 2 : 1$.

Show that A , N and B are on the same straight line.



STEP-BY-STEP SOLUTION:

1. Find $\vec{AN}$:

$$\vec{PQ} = -8\mathbf{p} + 6\mathbf{q}$$

$$\vec{PN} = \frac{1}{2}(-8\mathbf{p} + 6\mathbf{q}) = -4\mathbf{p} + 3\mathbf{q}$$

$$\vec{AP} = 2\mathbf{p} \text{ (since ratio is } 3 : 1, OP = 8\mathbf{p}\text{)}$$

$$\vec{AN} = \vec{AP} + \vec{PN} = 2\mathbf{p} - 4\mathbf{p} + 3\mathbf{q} = -2\mathbf{p} + 3\mathbf{q}$$

2. Find $\vec{AB}$:

$$\vec{AB} = \vec{AO} + \vec{OB}$$

$$\vec{AO} = -6\mathbf{p}$$

$$OB \text{ is } \frac{3}{2} \text{ of } OQ \text{ (ratio } 2 : 1\text{), so } \vec{OB} = 9\mathbf{q}$$

$$\vec{AB} = -6\mathbf{p} + 9\mathbf{q}$$

3. Compare vectors:

$$\vec{AN} = -2\mathbf{p} + 3\mathbf{q}$$

$$\vec{AB} = 3(-2\mathbf{p} + 3\mathbf{q})$$

Since $\vec{AB}$ is a scalar multiple of $\vec{AN}$ (specifically, $\vec{AB} = 3\vec{AN}$) and they share the common point A , they must lie on the same straight line.

Proven

3 Parallelogram Proofs

□ CORE CONCEPT & METHOD

In a parallelogram, opposite sides are equal and parallel. So if $\vec{OP} = \mathbf{p}$ and $\vec{OQ} = \mathbf{q}$, then $\vec{QR} = \mathbf{p}$ and $\vec{PR} = \mathbf{q}$. Use these parallel pathways to find your vectors!

COMMON MISTAKE: Check vertex order around the parallelogram carefully: in $OACB$, $\vec{OC} = \vec{OA} + \vec{OB}$ is the diagonal, whereas in $OABC$, $\vec{OB}$ is the diagonal.

MODEL WORKED EXAMPLES

EXAMPLE 3.1

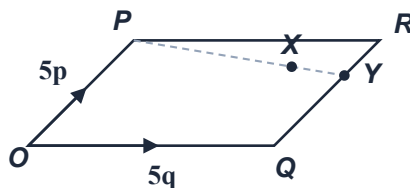
Non-Calculator

The diagram shows a parallelogram $OPRQ$. $\vec{OP} = 5\mathbf{p}$ and $\vec{OQ} = 5\mathbf{q}$.

X is the point on OR such that $OX : XR = 4 : 1$.

Y is the point on QR such that $QY : YR = 3 : 1$.

Show that P , X and Y are on the same straight line.



STEP-BY-STEP SOLUTION:

1. Find $\vec{PX}$:

$$\vec{OR} = \vec{OP} + \vec{PR} = 5\mathbf{p} + 5\mathbf{q}$$

$$\vec{OX} = \frac{4}{5}\vec{OR} = 4\mathbf{p} + 4\mathbf{q}$$

$$\vec{PX} = \vec{PO} + \vec{OX} = -5\mathbf{p} + 4\mathbf{p} + 4\mathbf{q} = -\mathbf{p} + 4\mathbf{q}$$

2. Find $\vec{PY}$:

Y is on QR ($QY : YR = 3 : 1$). $\vec{QR} = \vec{OP} = 5\mathbf{p}$.

$$\vec{QY} = \frac{3}{4}\vec{QR} = \frac{15}{4}\mathbf{p}$$

$$\vec{PY} = \vec{PQ} + \vec{QY} = (-5\mathbf{p} + 5\mathbf{q}) + \frac{15}{4}\mathbf{p}$$

$$\vec{PY} = -\frac{5}{4}\mathbf{p} + 5\mathbf{q}$$

3. Compare:

$$\vec{PX} = 1(-\mathbf{p} + 4\mathbf{q})$$

$$\vec{PY} = \frac{5}{4}(-\mathbf{p} + 4\mathbf{q})$$

Since $\vec{PY} = \frac{5}{4}\vec{PX}$ and they share point P , they are collinear.

Proven

4 Finding Unknown Extensions

CORE CONCEPT & METHOD

If you are told points are collinear, you can set an unknown line extension (like $\vec{OC}$) to be a multiple of its direction vector, e.g. $\vec{OC} = k\mathbf{y}$. Then equate the scalar multiples of $\mathbf{x}$ and $\mathbf{y}$ to solve for k !

COMMON MISTAKE: Never try to divide vectors — write the collinear vector in TWO independent ways using scalars (e.g. k and μ), then equate coefficients of $\mathbf{a}$ and $\mathbf{b}$ separately.

MODEL WORKED EXAMPLES

EXAMPLE 4.1

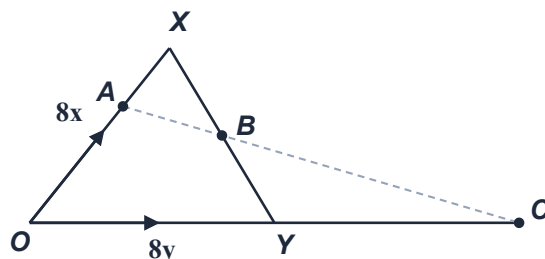
Non-Calculator

In the diagram, $\vec{OX} = 8\mathbf{x}$ and $\vec{OY} = 8\mathbf{y}$.

A is on OX such that $OA : AX = 3 : 1$.

B is the midpoint of XY .

The line OY is extended to point C . Given that A , B and C are on the same straight line, find $\vec{YC}$.



STEP-BY-STEP SOLUTION:

1. Find $\vec{AB}$:

$$\vec{XY} = -8\mathbf{x} + 8\mathbf{y}$$

$$\vec{XB} = \frac{1}{2}\vec{XY} = -4\mathbf{x} + 4\mathbf{y}$$

$$\vec{AX} = 2\mathbf{x}$$

$$\vec{AB} = \vec{AX} + \vec{XB} = 2\mathbf{x} - 4\mathbf{x} + 4\mathbf{y} = -2\mathbf{x} + 4\mathbf{y}$$

2. Express $\vec{AC}$:

$$\text{Let } \vec{OC} = k\mathbf{y}.$$

$$\vec{AC} = \vec{AO} + \vec{OC} = -6\mathbf{x} + k\mathbf{y}$$

3. Solve for k :

Since A , B , C are collinear, $\vec{AC}$ is a multiple of $\vec{AB}$.

$$\vec{AB} = -2\mathbf{x} + 4\mathbf{y}$$

$$\vec{AC} = -6\mathbf{x} + k\mathbf{y}$$

The x-component is multiplied by 3 (from -2 to -6).

So the y-component must also be multiplied by 3.

$$k = 4 \times 3 = 12.$$

$$\text{So } \vec{OC} = 12\mathbf{y}.$$

$$\vec{YC} = \vec{YO} + \vec{OC} = -8\mathbf{y} + 12\mathbf{y} = 4\mathbf{y}.$$

$$4\mathbf{y}$$

Probability Equations

1 Equations from Ratios

□ CORE CONCEPT & METHOD

If you are given a ratio like 2:1, use algebra! Let the number of marbles be $2x$ and x . The total is $3x$.

Then, set up your standard 'without replacement' probability fractions and solve the equation for x .

COMMON MISTAKE: When picking WITHOUT replacement, reduce BOTH the numerator (if same colour) AND the total denominator by 1 on the second pick — e.g. $\frac{2x}{5x} \times \frac{2x-1}{5x-1}$.

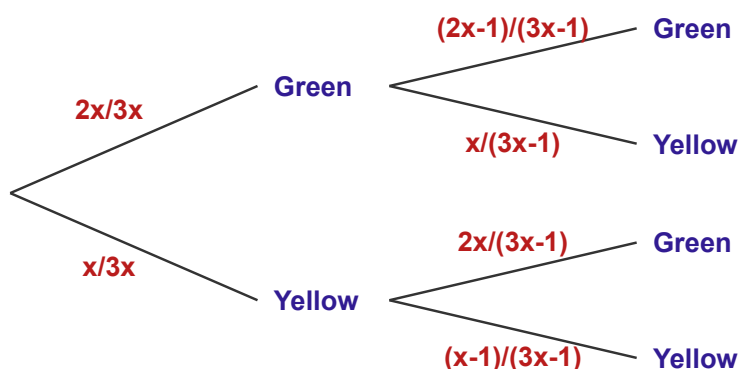
MODEL WORKED EXAMPLES

EXAMPLE 1.1

✎ Non-Calculator

A box contains green and yellow marbles in the ratio 2:1. Two marbles are taken **without replacement**.

The probability of picking two green marbles is $\frac{2}{5}$. How many marbles are in the box in total?



STEP-BY-STEP SOLUTION:

Step 1: Set up variables. Let the number of green marbles be $2x$ and yellow be x . Total = $3x$.

Step 2: Write the probability equation.

$$\frac{2x}{3x} \times \frac{2x-1}{3x-1} = \frac{2}{5}$$

Simplify $\frac{2x}{3x} = \frac{2}{3}$:

$$\frac{2}{3} \times \frac{2x-1}{3x-1} = \frac{2}{5}$$

Step 3: Cross-multiply and solve.

$$\begin{aligned} 2(2x-1) \times 5 &= 2 \times 3(3x-1) \\ 10(2x-1) &= 6(3x-1) \\ 20x-10 &= 18x-6 \\ 2x &= 4 \implies x = 2 \end{aligned}$$

Total marbles = $3x = 6$.

2 Equations With Replacement

CORE CONCEPT & METHOD

If the item is **replaced**, the fractions are identical. For example, if there are 4 purple and y white beads, the probability of picking two purples is:

$$\frac{4}{4+y} \times \frac{4}{4+y}$$

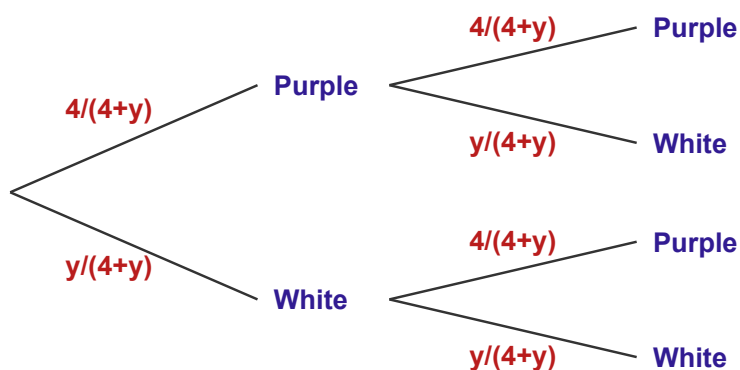
COMMON MISTAKE: For "one of each colour" (e.g. one red and one blue), remember to add BOTH orders (RB + BR) unless the question specifies "first red, then blue".

MODEL WORKED EXAMPLES

EXAMPLE 2.1

 **Non-Calculator**

A jar contains 4 purple and y white beads. A bead is picked, its colour noted, and then **replaced**. This is done twice. The probability of picking purple both times is $\frac{16}{81}$. Find y .



STEP-BY-STEP SOLUTION:

Because the bead is replaced, the total stays the same for both picks.

The probability of picking purple twice is:

$$\frac{4}{4+y} \times \frac{4}{4+y} = \frac{16}{81}$$

So:

$$\left(\frac{4}{4+y} \right)^2 = \frac{16}{81}$$

Take the square root of both sides:

$$\frac{4}{4+y} = \frac{4}{9}$$

Therefore $4 + y = 9$, giving $y = 5$.

3 Quadratic Probability Equations

CORE CONCEPT & METHOD

For 'without replacement' problems with an unknown total n , you will often need to expand brackets and form a quadratic equation (e.g., $n^2 - 13n - 30 = 0$), then factorise it to find n .

COMMON MISTAKE: Cross-multiply carefully, rearrange to $an^2 + bn + c = 0$ before factorising, and always reject negative or non-integer roots since a count n must be a positive integer.

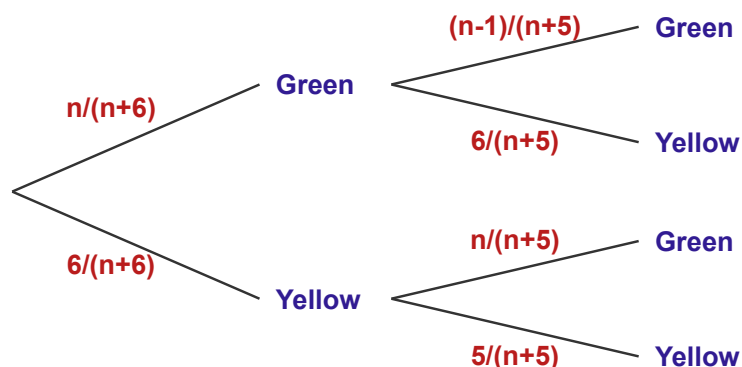
MODEL WORKED EXAMPLES

EXAMPLE 3.1



A box contains n green counters and 6 yellow counters. Two counters are drawn at random **without replacement**.

The probability that both counters are green is $\frac{1}{2}$. Show that $n^2 - 13n - 30 = 0$, and hence find the value of n .



STEP-BY-STEP SOLUTION:

First draw: Total counters = $n + 6$.

$$P(\text{Green}_1) = \frac{n}{n+6}$$

Second draw: One green counter is removed, so $n - 1$ greens remain out of a total of $n + 5$.

$$P(\text{Green}_2) = \frac{n-1}{n+5}$$

Equation:

$$\begin{aligned} \frac{n}{n+6} \times \frac{n-1}{n+5} &= \frac{1}{2} \\ \frac{n(n-1)}{(n+6)(n+5)} &= \frac{1}{2} \\ 2(n^2 - n) &= (n+6)(n+5) \end{aligned}$$

$$2n^2 - 2n = n^2 + 11n + 30$$

$$n^2 - 13n - 30 = 0$$

Factorise and solve:

$$(n-15)(n+2) = 0$$

Since $n > 0$, $n = 15$.